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Fecha: 28 de Abril

Auxiliar 6: Spline Cúbicos

Resumen Materia

1. **Spline Cúbica:** $S(x) = \{S_k(x)\ x \in [x_k, x_{k+1}]\ \ k \in \{0..n-1\}\}$ y cumple

   (a) i. Interpole: $S_k(x_k) = y_k$ — n+1 restricciones
       ii. Continua: $S_{k-1}(x_k) = S_k(x_k)$ — n-1 restricciones
       iii. 1° Derivada Continua: $S'_{k-1}(x_k) = S'_k(x_k)$ — n-1 restricciones
       iv. 2° Derivada Continua: $S''_{k-1}(x_k) = S''_k(x_k)$ — n-1 restricciones

2. **Condiciones de Borde:** Son 4n incógnitas y 4n-2 restricciones + 2 condiciones de borde

   (a) Spline Natural: $S''(x_0) = 0,\ S''(x_n) = 0$
   (b) Spline Extremos Constantes: $S''(x_0) = S''(x_1),\ S''(x_n) = S''(x_{n-1})$
   (c) Spline Valor Fijo: $S''(x_0) = \alpha,\ S''(x_n) = \beta$
   (d) Spline Sujeta: $S'(x_0) = f'(x_0),\ S'(x_n) = f'(x_n)$

3. **Construcción de la Spline:** se define $m_k = \frac{S''(x_k)}{2}$, $h_k = x_{k+1} - x_k$, $d_k = \frac{y_{k+1}-y_k}{h_k}$, $\mu_k = 3(d_k - d_{k-1})$ y se resuelve el sistema matricial de n+1 incógnitas

$$\begin{bmatrix} 1^\circ & \text{fila} & \text{condición} & \text{de} & \text{borde} & & & \\ h_0 & 2(h_0+h_1) & h_1 & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & h_2 & 2(h_2+h_3) & h_3 & 0 & \ldots & 0 \\ \vdots & 0 & 0 & \vdots & \vdots & \vdots & \ldots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & 0 \\ 0 & 0 & 0 & 0 & \ldots & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ N^\circ & \text{fila} & \text{condición} & \text{de} & \text{borde} & & & \end{bmatrix} \begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ \vdots \\ \vdots \\ \vdots \\ m_{n-1} \\ m_n \end{bmatrix} = \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \vdots \\ \vdots \\ \vdots \\ \mu_{n-1} \\ \mu_n \end{bmatrix}$$

   Ahora, se construye la Spline de la siguiente forma:
   $S_k(x) = S_{k,0} + S_{k,1}(x-x_k) + S_{k,2}(x-x_k)^2 + S_{k,3}(x-x_k)^3$ donde
   $S_{k,0} = y_k$ $\quad S_{k,1} = d_k - \frac{h_k(2m_k+m_{k+1})}{3}$ $\quad S_{k,2} = m_k$ $\quad S_{k,3} = \frac{m_{k+1}-m_k}{3h_k}$

Problemas

1. Por las fuertes lluvias que se pronostican para el mes de Mayo, se desean hacer arreglos en la carretera Autopista Central para evitar inundaciones, pero el Ingeniera que diseño la carretera era de la Universidad de las Américas, y perdió los planos. El gerente, desesperado, llama a un Ingeniero amigo suyo de Beauchef, y le plantea el problema. El Beauchefiano, que pasó Calculo Numérico, sabe muy bien que hacer. Le pide al gerente una tabla de datos tomados en los Portales de Peaje que indiquen tiempo y posición de un vehículo. La tabla que el gerente le entrega es la siguiente:

t	0	3	5	8	11
x	0	225	383	623	1001
y	0	9	25	64	121
y'	0				22

   Haga lo que un Ingeniero de Beauchef haría.

   Hint: Use Spline Cúbica y Polinomio de Newton

Respuestas

1. Primero se construiye un Spline cúbico $S_x(t)$ el cual depende de $S_i(t), i = 0, 1, 2, 3$, donde

$$S_x(t) = \begin{cases} S_0(t) = S_{0,0} + S_{0,1}(t-0) + S_{0,2}(t-0)^2 + S_{0,3}(t-0)^3 & t \in [0,3] \\ S_1(t) = S_{1,0} + S_{1,1}(t-3) + S_{1,2}(t-3)^2 + S_{1,3}(t-3)^3 & t \in [3,5] \\ S_2(t) = S_{2,0} + S_{2,1}(t-5) + S_{2,2}(t-5)^2 + S_{2,3}(t-5)^3 & t \in [5,8] \\ S_3(t) = S_{3,0} + S_{3,1}(t-8) + S_{3,2}(t-8)^2 + S_{3,3}(t-8)^3 & t \in [8,10] \end{cases}$$

La matriz es

$$\begin{bmatrix} 1^\circ & \text{Condicion} & \text{de} & \text{borde} & \\ h_0 & 2(h_0+h_1) & h_1 & 0 & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & 0 \\ 0 & 0 & h_2 & 2(h_2+h_3) & h_3 \\ 2^\circ & \text{Condicion} & \text{de} & \text{borde} & \end{bmatrix} \begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} \mu_0 \\ \mu_1 = 3(d_1 - d_0) \\ \mu_2 = 3(d_2 - d_1) \\ \mu_3 = 3(d_3 - d_2) \\ \mu_4 \end{bmatrix}$$

Los coeficientes son

$$\begin{array}{ll} h_0 = 3 - 0 = 3 & d_0 = \frac{225-0}{3} = 75 \\ h_1 = 5 - 3 = 2 & d_1 = \frac{383-225}{2} = 79 \\ h_2 = 8 - 5 = 3 & d_2 = \frac{623-383}{3} = 80 \\ h_3 = 11 - 8 = 3 & d_3 = \frac{1001-623}{3} = 126 \end{array}$$

Reemplazando en la matriz es

$$\begin{bmatrix} & & & & \\ 3 & 10 & 2 & 0 & 0 \\ 0 & 2 & 10 & 3 & 0 \\ 0 & 0 & 3 & 12 & 3 \\ & & & & \end{bmatrix} \begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} \mu_0 \\ 12 \\ 3 \\ 138 \\ \mu_4 \end{bmatrix}$$

C.B: Spline Natural

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 10 & 2 & 0 & 0 \\ 0 & 2 & 10 & 3 & 0 \\ 0 & 0 & 3 & 12 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ 3 \\ 138 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{116}{59} \\ -\frac{226}{59} \\ \frac{735}{59} \\ 0 \end{bmatrix}$$

$$S_{k,0} = y_k \qquad S_{k,1} = d_k - \frac{h_k(2m_k + m_{k+1})}{3} \qquad S_{k,2} = m_k \qquad S_{k,3} = \frac{m_{k+1} - m_k}{3h_k}$$

$$S_x(t) = \begin{cases} S_0(t) = 0 + \left(75 - \frac{3(2\times 0 + \frac{116}{59})}{3}\right)(t-0) + 0 \times (t-0)^2 + \frac{\frac{116}{59} - 0}{3\times 3}(t-0)^3 & t \in [0,3] \\ S_1(t) = 225 + \left(79 - \frac{2(2\times\frac{116}{59} - \frac{226}{59})}{3}\right)(t-3) + \frac{116}{59}(t-3)^2 + \frac{-\frac{226}{59} - \frac{116}{59}}{3\times 2}(t-3)^3 & t \in [3,5] \\ S_2(t) = 383 + \left(80 - \frac{3(2\times\left(-\frac{226}{59}\right) + \frac{735}{59})}{3}\right)(t-5) + \left(-\frac{226}{59}\right)(t-5)^2 + \frac{\frac{735}{59} + \frac{226}{59}}{3\times 3}(t-5)^3 & t \in [5,8] \\ S_3(t) = 623 + \left(126 - \frac{3(2\times\frac{735}{59} + 0)}{3}\right)(t-8) + \frac{735}{59} \times (t-8)^2 + \frac{0 - \frac{735}{59}}{3\times 3}(t-8)^3 & t \in [8,11] \end{cases}$$

$$S_x(t) = \begin{cases} \frac{4309}{59}t + \frac{116}{531}t^3 & \text{if} \quad 0 \le t \wedge t \le 3 \\ \frac{2422}{59}t + \frac{629}{59}t^2 - \frac{57}{59}t^3 + \frac{1887}{59} & \text{if} \quad 3 \le t \wedge t \le 5 \\ \frac{44\,116}{177}t - \frac{5483}{177}t^2 + \frac{961}{531}t^3 - \frac{167\,267}{531} & \text{if} \quad 5 \le t \wedge t \le 8 \\ \frac{2695}{59}t^2 - 364t - \frac{245}{177}t^3 + \frac{233\,695}{177} & \text{if} \quad 8 \le t \wedge t \le 11 \end{cases}$$

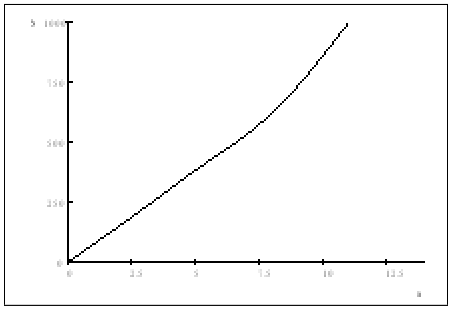
$$S(0) = 0, \quad S(3) = 225, \quad S(5) = 383, \quad S(8) = 623, \quad S(11) = 1001$$

$$D_x(t) = \begin{cases} \frac{116}{177}t^2 + \frac{4309}{59} & \text{if} \quad 0 \le t \wedge t \le 3 \\ \frac{1258}{59}t - \frac{171}{59}t^2 + \frac{2422}{59} & \text{if} \quad 3 \le t \wedge t \le 5 \\ -\frac{10\,966}{177}t + \frac{961}{177}t^2 + \frac{44\,116}{177} & \text{if} \quad 5 \le t \wedge t \le 8 \\ \frac{5390}{59}t - \frac{245}{59}t^2 - 364 & \text{if} \quad 8 \le t \wedge t \le 11 \end{cases}$$

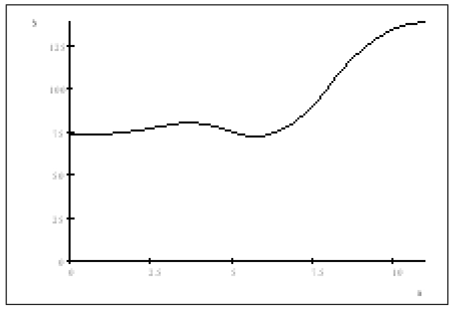
$$\begin{aligned} D(0) &= \frac{17\,807}{241} = 73.887\,966\,804\,979\,253\,112 \\ D(3) &= \frac{18\,611}{241} = 77.224\,066\,390\,041\,493\,776 \\ D(5) &= \frac{19\,359}{241} = 80.327\,800\,829\,875\,518\,672 \\ D(8) &= \frac{18\,804}{241} = 78.024\,896\,265\,560\,165\,975 \\ D(11) &= \frac{87\,909}{1205} = 72.953\,526\,970\,954\,356\,846 \end{aligned}$$

$$F_x(t) = \begin{cases} \frac{232}{177}t & \text{if} \quad 0 \le t \wedge t \le 3 \\ -\frac{342}{59}t + \frac{1258}{59} & \text{if} \quad 3 \le t \wedge t \le 5 \\ \frac{1922}{177}t - \frac{10\,966}{177} & \text{if} \quad 5 \le t \wedge t \le 8 \\ -\frac{490}{59}t + \frac{5390}{59} & \text{if} \quad 8 \le t \wedge t \le 11 \end{cases}$$

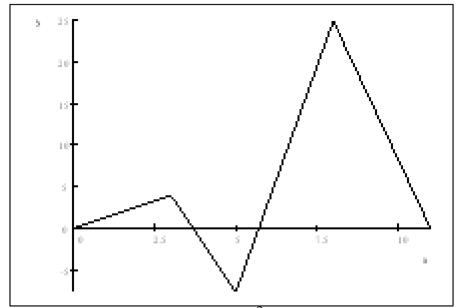
$$\begin{aligned} F(0) &= 0 \\ F(3) &= \frac{232}{59} = 3.932\,203\,389\,830\,508\,474\,6 \\ F(5) &= -\frac{452}{59} = -7.661\,016\,949\,152\,542\,372\,9 \\ F(8) &= \frac{1470}{59} = 24.915\,254\,237\,288\,135\,593 \\ F(11) &= 0 \end{aligned}$$



$S_x(t)$



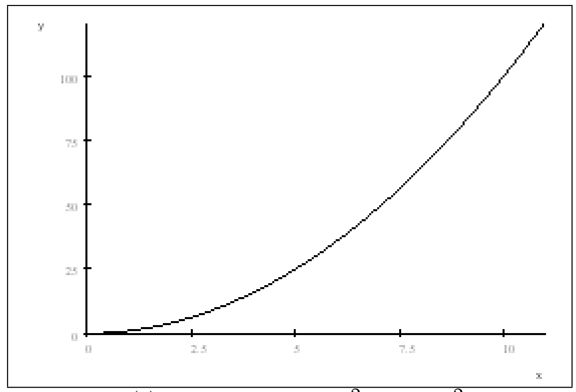
$D_x(t) = \frac{dS_x(t)}{dt}$



$F_x(t) = \frac{d^2S_x(t)}{dt^2}$

2. Luego, para la coordenada y hacemos una interpolación de Newton

0	0	$y'(0) = 0$	$\frac{3-0}{3-0} = 1$	0	0	0	0
0	0	$\frac{9-0}{3-0} = 3$	$\frac{8-3}{5-0} = 1$	0	0	0	
3	9	$\frac{25-9}{5-3} = 8$	$\frac{13-8}{8-3} = 1$	0	0		
5	25	$\frac{64-25}{8-5} = 13$	$\frac{19-13}{11-5} = 1$	0			
8	64	$\frac{121-64}{11-8} = 19$	$\frac{22-19}{11-8} = 1$				
11	121	$y'(11) = 22$					
11	121						



$y(t) = 0 + 0 \times t + 1t^2 + 0 = t^2$

despejando $t \Longrightarrow t = \sqrt{y}$

reemplazando en $S_x(t)$

$$S_x(t) = S_x(\sqrt{y}) = \begin{cases} \frac{4309}{59}\sqrt{y} + \frac{116}{531}\sqrt{y}^3 & \text{if} \quad 0 \leq \sqrt{y} \wedge \sqrt{y} \leq 3 \\ \frac{2422}{59}\sqrt{y} + \frac{629}{59}\sqrt{y}^2 - \frac{57}{59}\sqrt{y}^3 + \frac{1887}{59} & \text{if} \quad 3 \leq \sqrt{y} \wedge \sqrt{y} \leq 5 \\ \frac{44\,116}{177}\sqrt{y} - \frac{5483}{177}\sqrt{y}^2 + \frac{961}{531}\sqrt{y}^3 - \frac{167\,267}{531} & \text{if} \quad 5 \leq \sqrt{y} \wedge \sqrt{y} \leq 8 \\ \frac{2695}{59}\sqrt{y}^2 - 364\sqrt{y} - \frac{245}{177}\sqrt{y}^3 + \frac{233\,695}{177} & \text{if} \quad 8 \leq \sqrt{y} \wedge \sqrt{y} \leq 11 \end{cases}$$

transformando en una función que depende de y :

$$\Longrightarrow S_x(y) = \begin{cases} \frac{4309}{59}\sqrt{y} + \frac{116}{531}\sqrt{y}^3 & \text{if} \quad 0 \leq y \wedge y \leq 9 \\ \frac{2422}{59}\sqrt{y} + \frac{629}{59}y - \frac{57}{59}\sqrt{y}^3 + \frac{1887}{59} & \text{if} \quad 9 \leq y \wedge y \leq 25 \\ \frac{44\,116}{177}\sqrt{y} - \frac{5483}{177}y + \frac{961}{531}\sqrt{y}^3 - \frac{167\,267}{531} & \text{if} \quad 25 \leq y \wedge y \leq 64 \\ \frac{2695}{59}y - 364\sqrt{y} - \frac{245}{177}\sqrt{y}^3 + \frac{233\,695}{177} & \text{if} \quad 64 \leq y \wedge y \leq 121 \end{cases}$$

Graficando en un plano xy se obtiene el plano pedido.