Una placa delgada elíptica de semi-eje a,b (a≥b) puede moverse libremente en torno a su centro que está fijo. Es colocada en mov. dándole una velocidad angular de magnitud "n" entorno de un eje sobre su plano, igualmente inclinado respecto a ambos semiejes de la elipse. Dem. que el eje instantáneo de rotación volverá a estar sobre el plano de la elipse después de un tpo:

$$T = 2\int_0^{\lambda} \frac{1}{\sqrt{\lambda^4 - x^2}}\,dx = \frac{2}{\lambda}\int_0^{1} \frac{1}{\sqrt{1-x'^4}}\,dx'$$

$$y \;\; \lambda^2 = \frac{\Omega^2}{2} = \frac{(a^2-b^2)n}{(a^2+b^2)} \qquad = \frac{1}{\lambda}\int_0^{\pi/2} \frac{d\phi}{\sqrt{\cos\phi}}$$

Solución: Sean I_x, I_y, I_z momentos de inercia en los ejes principales.

$\Rightarrow$ ecs. de euler:

$$I_x\dot{\omega}_x + \omega_z\omega_y(I_z - I_y) = 0$$
$$I_y\dot{\omega}_y + \omega_x\omega_z(I_x - I_z) = 0$$
$$I_z\dot{\omega}_z + \omega_y\omega_x(I_y - I_x) = 0$$

pero como es placa plana $\Rightarrow I_z = I_x + I_y$

$$\Rightarrow \quad I_x\dot{\omega}_x + \omega_z\omega_y(I_x + \cancel{I_y} - \cancel{I_y}) = 0$$
$$I_y\dot{\omega}_y + \omega_x\omega_z(I_x - I_x - I_y) = 0$$
$$I_z\dot{\omega}_z + \omega_y\omega_x(I_y - I_x) = 0$$

$$\Rightarrow \quad I_x\dot{\omega}_x + I_x\omega_z\omega_y = 0 \Rightarrow \dot{\omega}_x + \omega_z\omega_y = 0 \quad (1)$$
$$I_y\dot{\omega}_y - I_y\omega_x\omega_z = 0 \Rightarrow \dot{\omega}_y - \omega_x\omega_z = 0 \quad (2)$$

$$\Rightarrow (I_x + I_y)\dot{\omega}_z + \omega_y \omega_x (I_y - I_x) = 0 \qquad (3)$$

$$(1) + (2) \Rightarrow \omega_x^2 + \omega_y^2 = cte.$$

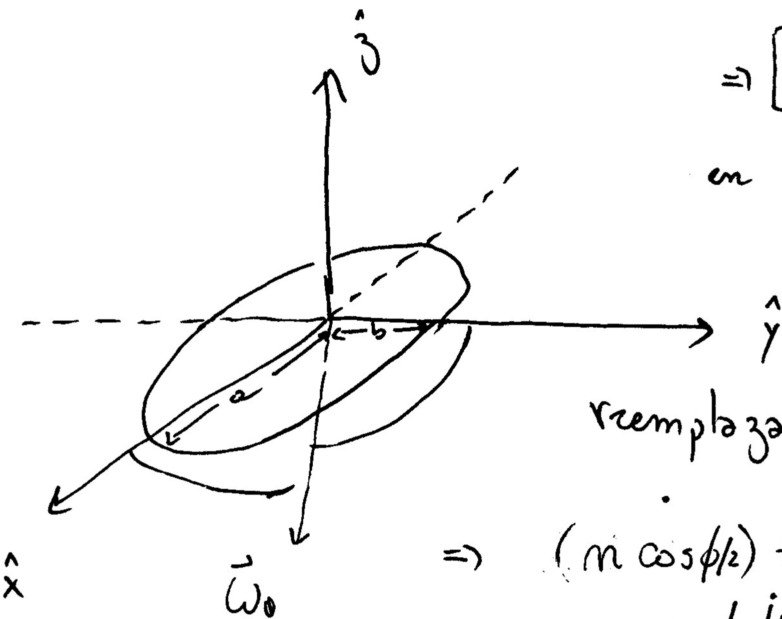
ya que $(1)\cdot\omega_x + (2)\cdot\omega_y = \dot{\omega}_x \omega_x + \dot{\omega}_y \omega_y = 0$

$$\Rightarrow \frac{d}{dt}\left(\frac{\omega_x^2}{2} + \frac{\omega_y^2}{2}\right) = 0$$

$$\Rightarrow \boxed{\omega_x^2 + \omega_y^2 = cte.}$$

y $\vec{\omega}_0 = \omega_x^0 \hat{x} + \omega_y^0 \hat{y}$ con $m = \sqrt{(\omega_x^0)^2 + (\omega_y^0)^2}$

$$\Rightarrow \boxed{\omega_x^2 + \omega_y^2 = m^2} \quad \forall t$$



en particular supongo

$$\omega_x = m\cdot\cos(\phi(t)/2)$$
$$\omega_y = m\cdot \text{sen}(\phi(t)/2)$$

reemplazando en (1)

$$\Rightarrow (m\cos\phi/2)^{\cdot} + \omega_z m\,\text{sen}\,\phi = 0$$
$$\Rightarrow -m\,\text{sen}\,\phi/2\,\dot{\phi}/2 + \omega_z m\,\text{sen}\,\phi/2 = 0$$
$$\Rightarrow \boxed{\omega_z = \dot{\phi}/2}$$

y en (3):

$$(I_x + I_y)\frac{\ddot{\phi}}{2} + m^2\,\text{sen}\frac{\phi}{2}\cos\frac{\phi}{2}(I_y - I_x) = 0$$

$$\Rightarrow (I_x + I_y)\frac{\ddot{\phi}}{2} + m^2\frac{\text{sen}\,\phi}{2}(I_y - I_x) = 0$$

$$\Rightarrow (I_x + I_y)\ddot{\phi} + m^2(I_y - I_x)\,\text{sen}\,\phi = 0$$

con $\dot{\phi}(0) = 0 \quad (\omega_z(0) = 0)$

$\phi(0) = \pi/2$

$$\Rightarrow \ddot{\phi} + \frac{m^2(I_y - I_x)}{(I_x + I_y)}\,\mathrm{sen}\,\phi = 0$$

$$\Rightarrow \frac{d\dot{\phi}}{dt} = \frac{d\dot{\phi}}{d\phi}\cdot\frac{d\phi}{dt} = -\frac{m^2(I_y - I_x)}{(I_x + I_y)}\,\mathrm{sen}\,\phi$$

$$\Rightarrow d\dot{\phi}\,\dot{\phi} = -\frac{m^2(I_y - I_x)}{(I_x + I_y)}\,\mathrm{sen}\,\phi\, d\phi \quad /\int$$

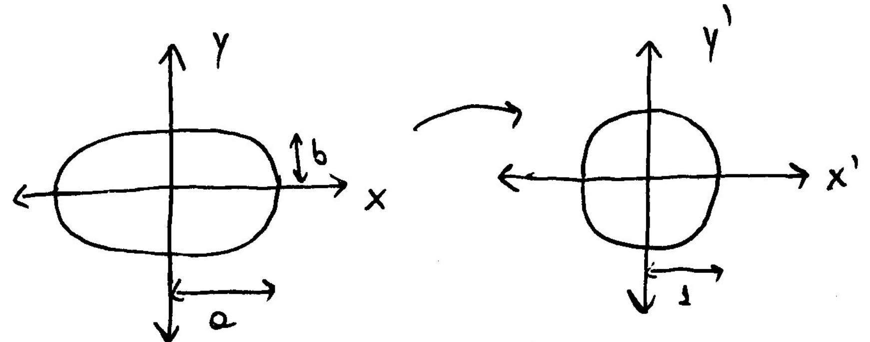
$$\frac{\dot{\phi}^2}{2} = +m^2\frac{(I_y - I_x)}{(I_x + I_y)}\cos\phi\,\Big|_{\pi/2}^{\phi}$$

$$\Rightarrow \frac{\dot{\phi}^2}{2} = m^2\frac{(I_y - I_x)}{(I_x + I_y)}\cos\phi$$

Antes de continuar:

$I_x = \sigma\int y^2\,dx\,dy$ con $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ y $\sigma = \frac{M}{\pi ab}$

$I_y = \sigma\int x^2\,dx\,dy$



$x = ax'$
$y = by'$

$$\Rightarrow I_x = \frac{M}{\pi ab}\int b^2 y'^2 \cdot a\,dx'\,b\,dy'$$

$$= \frac{Mb^2}{\pi}\overbrace{\int y'^2\,dx'\,dy'}^{I_1}$$

De igual forma:

$$I_y = \frac{Ma^2}{\pi} \underbrace{\int x'^2 \, dx' dy'}_{I_2}$$

Notar que: $I_1 + I_2 = \int x'^2 + y'^2 \, dx' dy' = I$ $\left(\underset{\text{por simetría}}{I_1 = I_2} \right) \Rightarrow I_1 = \frac{I}{2}$

$$I_2 = \frac{I}{2}$$

$$\Rightarrow I = \int_0^{2\pi}\int_0^1 r'^2 \, r' \, dr' d\theta'$$

$$= 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$$

$$\Rightarrow I_x = \frac{Mb^2}{4} \; ; \; I_y = \frac{Ma^2}{4}$$

$$\Rightarrow \frac{\dot{\phi}^2}{2} = \frac{m^2 \cancel{(M/4)} (a^2 - b^2) \cos\phi}{\cancel{(M/4)} (a^2 + b^2)} \quad , \text{pero}$$

$$= m^2 \frac{(a^2 - b^2)}{(a^2 + b^2)} \cos\phi \qquad \text{sea } \Omega^2 = \frac{m^2 (a^2 - b^2)}{(a^2 + b^2)}$$

$$\Rightarrow \dot{\phi}^2 = 2\Omega^2 \cos\phi$$

$$\Rightarrow \dot{\phi} = \Omega \sqrt{2\cos\phi}$$

$$\frac{d\phi}{\sqrt{\cos\phi}} = dt \cdot \Omega \cdot \sqrt{2}$$

tpo en que vuelve a pasar

$$\Rightarrow \dot{\phi} = 0 \Rightarrow \phi = \frac{\pi}{2}, \frac{3\pi}{2}$$

pero $\sqrt{\cos\phi} \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$

$$\Rightarrow \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\sqrt{\cos\phi}} = \Omega\sqrt{2}\,T$$

$$\Rightarrow 2\int_{0}^{\pi/2} \frac{d\phi}{\sqrt{\cos\phi}} = \Omega\sqrt{2}\,T$$

$$\Rightarrow T = \frac{2}{\Omega\sqrt{2}} \int_{0}^{\pi/2} \frac{d\phi}{\sqrt{\cos\phi}}$$