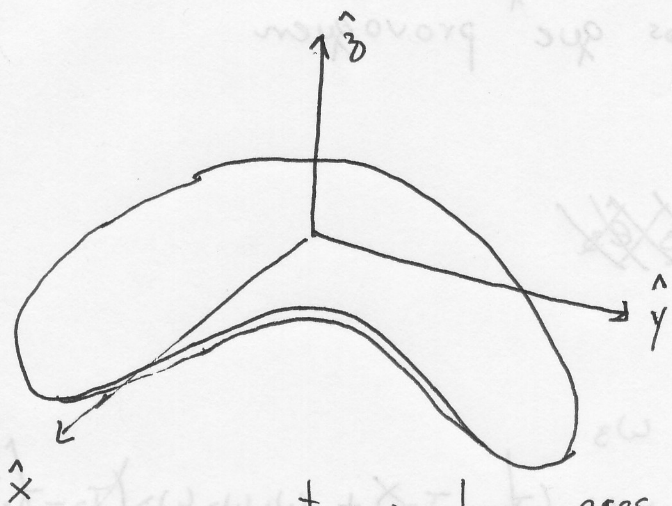


Un boomerang plano se lanza al aire rotando inicialmente con velocidad angular $\vec{\omega}(t=0)$ tq:

$$\vec{\omega}_0 = \omega_0 \hat{z} + \omega_0 \sqrt{3}\, \hat{y}$$

Los momentos de inercia respecto a los ejes principales $\hat{x}$ e $\hat{y}$ son conocidos y valen $I_x = 2 I_0$, $I_y = I_0$

$$I_z = I_x + I_y = 3 I_0$$

Calcule $\vec{\omega}(t)$.

<u>Ecs. de Euler:</u>

$$I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2) = \tau_{e1\,(CM)}^{ext}$$
$$I_2 \dot{\omega}_2 + \omega_3 \omega_1 (I_1 - I_3) = \tau_{e2\,(CM)}^{ext}$$
$$I_3 \dot{\omega}_3 + \omega_1 \omega_2 (I_2 - I_1) = \tau_{e3\,(CM)}^{ext}$$

Cuando el cuerpo no está bajo la acción de fuerzas externas

$$\Rightarrow \quad I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2) = 0 \qquad (1)$$
$$I_2 \dot{\omega}_2 + \omega_3 \omega_1 (I_1 - I_3) = 0 \qquad (2)$$
$$I_3 \dot{\omega}_3 + \omega_1 \omega_2 (I_2 - I_1) = 0 \qquad (3)$$

$$\Rightarrow \quad 2 I_0 \dot{\omega}_x + \omega_y \omega_z (3I_0 - I_0) = 0 \quad \Rightarrow \quad 2\dot{\omega}_x + 2\omega_y \omega_z = 0$$
$$I_0 \dot{\omega}_y + \omega_z \omega_x (2 I_0 - 3 I_0) = 0 \quad \Rightarrow \quad \dot{\omega}_y - \omega_z \omega_x = 0$$
$$3 I_0 \dot{\omega}_z + \omega_x \omega_y (I_0 - 2 I_0) = 0 \quad \Rightarrow \quad 3\dot{\omega}_z - \omega_x \omega_y = 0$$

sistema de 3 ecs. y 3 incógnitas

Antes de ponerse a integrar veamos qué ocurre con un sólido libre de fuerzas externas que provoquen torque.

$$\cancel{\vec{L}^{cm} = I_1 \omega_1 \hat{e}_1 + I_2 \omega_2 \hat{e}_2 + I_3 \omega_3 \hat{e}_3}$$

$$(1)\cdot\omega_1 + (2)\cdot\omega_2 + (3)\cdot\omega_3$$

$$\Rightarrow I_1 \omega_1 \dot{\omega}_1 + I_2 \omega_2 \dot{\omega}_2 + I_3 \omega_3 \dot{\omega}_3$$
$$+ \omega_2\omega_1\omega_3 \cancel{(I_3 - I_2)} + \omega_1\omega_2\omega_3 \cancel{(I_1 - I_3)} + \omega_1\omega_2\omega_3 \cancel{(I_2 - I_1)} = 0$$

$$\Rightarrow I_1 \frac{d}{dt}\left(\frac{\omega_1^2}{2}\right) + I_2 \frac{d}{dt}\left(\frac{\omega_2^2}{2}\right) + I_3 \frac{d}{dt}\left(\frac{\omega_3^2}{2}\right) = 0$$

$$\Rightarrow I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = cte.$$

Cuando no hay torque:

$$\Rightarrow \vec{L}^{ep} = cte.$$

$$\Rightarrow I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = cte.$$

En nuestro caso:

- $2 I_0 \omega_1^2 + I_0 \omega_2^2 + 3 I_0 \omega_3^2 = 2 I_0 \cdot \cancel{0}^{\,0} + I_0 \omega_0^2 \cdot 3 + 3 I_0 \omega_0^2$
  $= 6 I_0 \omega_0^2$

- $4 I_0^2 \omega_1^2 + I_0^2 \omega_2^2 + 9 I_0^2 \omega_3^2 = I_0^2 \cdot 3\omega_0^2 + 9 I_0^2 \omega_0^2$
  $= 12 \omega_0^2$

$$\Rightarrow 2\omega_1^2 + \omega_2^2 + 3\omega_3^2 = 6\omega_0^2 \quad (1)$$
$$4\omega_1^2 + \omega_2^2 + 9\omega_3^2 = 12\omega_0^2 \quad (2)$$

(2) - (1):

$$2\omega_x^2 + 6\omega_z^2 = 6\omega_0^2$$

$$\Rightarrow \omega_z = \sqrt{\omega_0^2 - \frac{\omega_x^2}{3}}$$

$$\frac{d\omega_x}{\left(1 - \frac{\omega_x^2}{3\omega_0^2}\right)\omega_0^2}$$

C.V: $u = \frac{\omega_x}{\sqrt{3}\,\omega_0}$

$$\Rightarrow du = \frac{d\omega_x}{\sqrt{3}\,\omega_0}$$

$$\Rightarrow I = \int_0^{u\sqrt{3}\,\omega_0} \frac{\sqrt{3}\,\cancel{\omega_0}\, du}{(1-u^2)\,\omega_0^2}$$

(2) - 2·(1):

$$-\omega_y^2 + 3\omega_z^2 = 0$$

$$\Rightarrow \omega_y = \sqrt{3}\,\omega_z = \sqrt{3}\sqrt{\omega_0^2 - \frac{\omega_x^2}{3}}$$

(1) $\Rightarrow$ $\dot{\omega}_x = -\omega_y\omega_z$

$$\Rightarrow \dot{\omega}_x = -\sqrt{3}\left(\omega_0^2 - \frac{\omega_x^2}{3}\right)$$

$$\Rightarrow \frac{d\omega_x}{dt} = -\sqrt{3}\left(\omega_0^2 - \frac{\omega_x^2}{3}\right)$$

$$\Rightarrow \frac{d\omega_x}{\left(\omega_0^2 - \frac{\omega_x^2}{3}\right)} = -\sqrt{3}\,dt \quad \Big/ \int_{\omega_x(0)}^{\omega_x(t)} = \int_0^t$$

$$\frac{\cancel{\sqrt{3}}}{\omega_0}\,\operatorname{arctanh} u\,\Big|_0^u = -\cancel{\sqrt{3}}\cdot t$$

$$\Rightarrow \operatorname{arctanh} u = -\omega_0 \cdot t$$

$$\Rightarrow u = \tanh(-\omega_0 \cdot t)$$

$$\Rightarrow \boxed{\omega_x = -\sqrt{3}\,\omega_0 \cdot \tanh(\omega_0 \cdot t)}$$

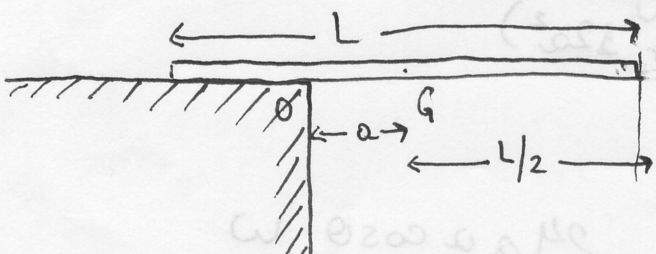
$$M\frac{d\vec{V}_G}{dt} = \vec{F}^{ext}$$

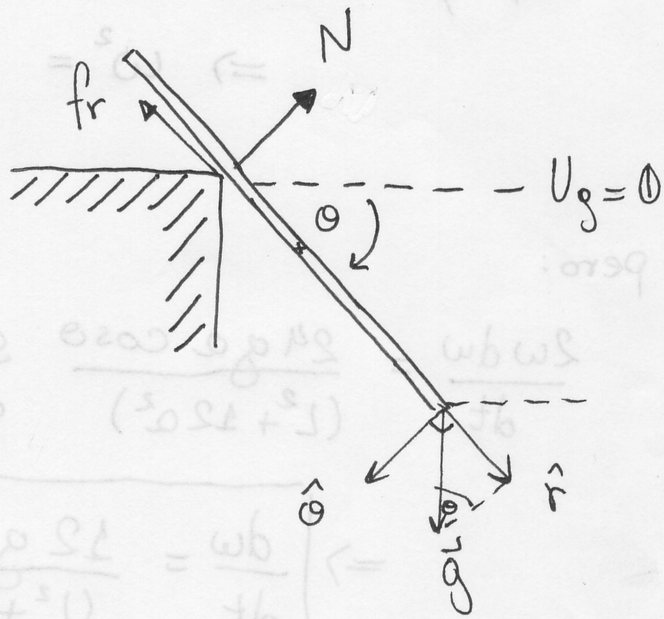
$$\vec{L}_s = M\vec{R}_G \times \vec{V}_G + \vec{L}'$$

$$\Rightarrow \frac{d\vec{L}_s}{dt} = \vec{\tau}^{ext}$$

$$\Rightarrow \frac{d}{dt}\left(M\vec{R}_G \times \vec{V}_G + \vec{L}'\right) = \vec{\tau}^{ext}$$

Problema 1





$$T = \frac{1}{2}M\vec{V}_0^2 + M\vec{V}_0\vec{V}_{G c/r O} + \frac{1}{2}I_0\vec{\omega}^2$$

$$\vec{V}_0 = \vec{0} \Rightarrow T = \frac{1}{2}I_0\vec{\omega}^2$$

$$I_{33} = \frac{1}{3}ML^2$$

pero $I_{cm} = \frac{1}{3}ML^2 - M\left(\frac{L}{2}\right)^2 = \frac{1}{12}ML^2$

$$\Rightarrow I_0 = \frac{1}{12}ML^2 + Ma^2$$

$$\Rightarrow T = \frac{1}{2} I_0 \omega^2 = \frac{1}{2}\left(\frac{1}{12} ML^2 + Ma^2\right)\omega^2$$

$$= \frac{M}{24}\left(L^2 + 12a^2\right)\omega^2$$

$$\Rightarrow E = \frac{M}{24}\left(L^2+12a^2\right)\omega^2 - Mga\,\text{sen}\,\theta$$

Por cond. iniciales:

$$E_0 = 0$$

$$\Rightarrow \cancel{M}\left(L^2+12a^2\right)\omega^2 = 24\cancel{M}ga\,\text{sen}\,\theta$$

$$\Rightarrow \omega^2 = \frac{24ga\,\text{sen}\,\theta}{(L^2+12a^2)}$$

pero:

$$\frac{2\omega\, d\omega}{dt} = \frac{24ga\cos\theta}{(L^2+12a^2)}\frac{d\theta}{dt} = \frac{24ga\cos\theta}{(L^2+12a^2)}\,\omega$$

$$\Rightarrow \boxed{\frac{d\omega}{dt} = \frac{12ga\cos\theta}{(L^2+12a^2)}}$$

$\hat{r})$ $$Mg\,\text{sen}\,\theta - f_r = M(\ddot{r} - r\dot{\theta}^2) = -Mr\omega^2$$

$$\Rightarrow Mg\,\text{sen}\,\theta - f_r = -Mr\omega^2$$

$\hat{\theta})$ $$Mg\cos\theta - N = M(2\dot{r}\dot{\theta} + r\ddot{\theta})$$

$$\boxed{\dot{\theta} = \omega}$$

$$\Rightarrow N = Mg\cos\theta' - r\ddot{\theta}M$$

$$N = Mg\cos\theta' - Mr\dot{\omega}$$

$$N = Mg\cos\theta' - \frac{M \cdot \cancel{a} \cdot 12ga\cos\theta'}{(L^2 + 12a^2)}$$

$$= Mg\cos\theta'\left(\frac{L^2 + \cancel{12a^2} - \cancel{12a^2}}{L^2 + 12a^2}\right)$$

$$\boxed{N = Mg\cos\theta' \frac{L^2}{(L^2 + 12a^2)}}$$

pero además:

$$\boxed{f_r = Mg\sin\theta' + Ma\omega^2}$$

$$f_r = Mg\sin\theta' + \frac{Ma\,24\,g\,a\sin\theta'}{(L^2+12a^2)}$$

$$= Mg\sin\theta'\left(\frac{L^2+12a^2+24a^2}{L^2+12a^2}\right)$$

$$\Rightarrow \boxed{f_r = Mg\sin\theta'\left(\frac{L^2+36a^2}{L^2+12a^2}\right)}$$

$$\Rightarrow \cancel{Mg}\sin\theta'\left(\frac{L^2+36a^2}{\cancel{L^2+12a^2}}\right) = \mu\,\cancel{Mg}\cos\theta'\,\frac{L^2}{\cancel{L^2+12a^2}}$$

$$\Rightarrow \boxed{\tan\theta' = \frac{\mu\,L^2}{(L^2+36a^2)}}$$