Encuentre las ecs. de mov. del sistema descrito por el Lagrangeano:

$$L = ml^2\left(\frac{-\theta\ddot{\theta}}{2} + \frac{\operatorname{sen}^2\theta\,\dot{\phi}^2}{2}\right) + mgl\cos\theta$$

¿Se pueden aplicar las ecs. de Euler-Lagrange para este sistema?

Resp: Sí, pero las ecs. que provienen de la acción:

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, \ddot{q}_i)\,dt$$

¿Cuáles son las ecuaciones?

Resp: Hay que deducirlas $\ddot{\smile}$

Entonces:

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, \ddot{q}_i)\,dt$$

$$\left\{ \delta S = \int_{t_1}^{t_2} L(q_i+\delta q_i, \dot{q}_i+\delta\dot{q}_i, \ddot{q}_i+\delta\ddot{q}_i)\,dt - \int_{t_1}^{t_2} L(q_i, \dot{q}_i, \ddot{q}_i)\,dt \right.$$

Principio Variacional $\Rightarrow \delta S = 0$

$$\delta S = \delta\int_{t_1}^{t_2} L(q_i, \dot{q}_i, \ddot{q}_i)\,dt = \int_{t_1}^{t_2} \delta L(q_i, \dot{q}_i, \ddot{q}_i)\,dt$$

$$= \int_{t_1}^{t_2} \sum_i \underbrace{\left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i + \frac{\partial L}{\partial \ddot{q}_i}\delta\ddot{q}_i\right)}_{(*)}\,dt$$

$$(*) \Rightarrow \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i = \frac{\partial L}{\partial \dot{q}_i}\dot{(\delta q_i)} = \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i$$

$$\frac{\partial L}{\partial \ddot{q}_i}\delta\ddot{q}_i = \frac{\partial L}{\partial \ddot{q}_i}\frac{d}{dt}(\delta\dot{q}_i) = \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_i}\delta\dot{q}_i\right) - \overbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\delta\dot{q}_i}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_i}\delta\dot{q}_i\right) - \left[\frac{d}{dt}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\delta q_i\right) - \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\delta q_i\right]$$

$$\Rightarrow \delta S = \int_{t_1}^{t_2} \sum_i \left\{ \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot q_i}\right) \right\} \delta q_i + \int_{t_1}^{t_2} \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\,\delta q_i + \frac{\partial L}{\partial \ddot q_i}\,\delta \dot q_i - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot q_i}\right)\delta q_i \right) dt$$

$$= \int_{t_1}^{t_2} \sum_i \left\{ \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot q_i}\right) \right\} \delta q_i$$

$$+ \sum_i \left(\frac{\partial L}{\partial \dot q_i}\,\delta q_i + \frac{\partial L}{\partial \ddot q_i}\,\delta \dot q_i \; \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot q_i}\right)\delta q_i \right)\Bigg|_{t_1}^{t_2}$$

Queremos $\delta S = 0$ y hemos variado a extremo fijo, i.e.:

$$\delta q_i(t_1) = \delta q_i(t_2) = \delta \dot q_i(t_1) = \delta \dot q_i(t_2) = 0$$

$$\Rightarrow \delta S = \int_{t_1}^{t_2} \sum_i \left\{ \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot q_i}\right) \right\} \delta q_i \, dt$$

$$\Rightarrow \boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot q_i}\right) = 0}$$

Ecuación de Euler-Lagrange $\forall i$

Entonces:

para θ

$$\frac{\partial L}{\partial \theta} = m l^2 \operatorname{sen}\theta \cos\theta \, \dot\phi^2 - m g l \operatorname{sen}\theta - \frac{m l^2 \ddot\theta}{2}$$

$$\frac{\partial L}{\partial \ddot\theta} = \frac{-m l^2 \theta}{2} \Rightarrow \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot\theta}\right) = \frac{-m l^2 \ddot\theta}{2}$$

$$\Rightarrow m l^2 \operatorname{sen}\theta \cos\theta \, \dot\phi^2 - m g l \operatorname{sen}\theta = m l^2 \ddot\theta$$

$$\Rightarrow \boxed{\ddot\theta = -\frac{g}{l} \operatorname{sen}\theta + \operatorname{sen}\theta \cos\theta \, \dot\phi^2}$$

Para ϕ

$$\frac{\partial L}{\partial \phi} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0$$

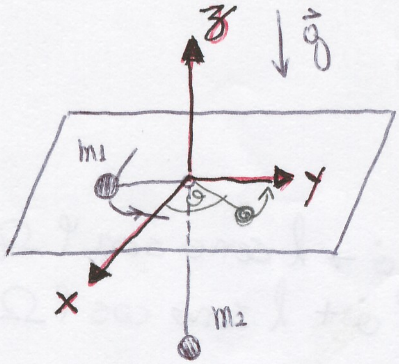
$$\frac{\partial L}{\partial \dot{\phi}} = 0$$

$$\Rightarrow \frac{\partial L}{\partial \dot{\phi}} = cte = m l^2 \operatorname{sen}^2\theta \, \dot{\phi} = P_\phi \quad \left(\text{momentum generalizado}\right)$$

$$\Rightarrow \dot{\phi} = \frac{P_\phi}{m l^2 \operatorname{sen}^2\theta}$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{l}\operatorname{sen}\theta + \operatorname{sen}\theta\cos\theta \, \frac{P_\phi^2}{m^2 l^4 \operatorname{sen}^4\theta}$$

$$\Rightarrow \boxed{\ddot{\theta} = -\frac{g}{l}\operatorname{sen}\theta + \frac{P_\phi^2 \cdot \cos\theta}{m^2 l^4 \operatorname{sen}^3\theta}}$$



$$x_1(r,\theta) = r\cos\theta \Rightarrow \dot{x}_1(r,\theta) = \dot{r}\cos\theta - r\,\text{sen}\theta\,\dot{\theta}$$
$$y_1(r,\theta) = r\,\text{sen}\theta \Rightarrow \dot{y}_1(r,\theta) = \dot{r}\,\text{sen}\theta + r\cos\theta\,\dot{\theta}$$
$$z_1(r,\theta) = 0 \Rightarrow \dot{z}_1(r,\theta) = 0$$

$$x_2(r,\theta) = 0 \Rightarrow \dot{x}_2(r,\theta) = 0$$
$$y_2(r,\theta) = 0 \Rightarrow \dot{y}_2(r,\theta) = 0$$
$$z_2(r,\theta) = -(l-r) \Rightarrow \dot{z}_2(r,\theta) = \dot{r}$$

$$\mathcal{L}(r,\dot{r},\theta,\dot{\theta}) = \frac{m_1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{m_2}{2}\dot{r}^2 + m_2 g(l-r)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m_1\dot{r} + m_2\dot{r} \Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = (m_1+m_2)\ddot{r}$$

$$\frac{\partial \mathcal{L}}{\partial r} = m_1 r\dot{\theta}^2 - m_2 g$$

$$\Rightarrow (m_1+m_2)\ddot{r} = m_1 r\dot{\theta}^2 - m_2 g$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m_1 r^2\dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = 2m_1 r\dot{r}\dot{\theta} + m_1 r^2\ddot{\theta} = 0$$

$$\Rightarrow m_1 r^2\dot{\theta} = cte$$
$$\Rightarrow l = cte.$$

$$\Rightarrow \dot{\theta} = \frac{l}{m_1 r^2} \Rightarrow \dot{\theta}^2 = \frac{l^2}{m_1^2 r^4}$$

$$\Rightarrow (m_1+m_2)\ddot{r} = m_1 r\frac{l^2}{m_1^2 r^4} - m_2 g$$

$$(m_1+m_2)\ddot{r} = \frac{l^2}{m_1 r^3} - m_2 g$$

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