$$\overset{[cm]}{L} = \frac{\overset{[cm]}{I} + ml^2}{2}\left(\dot{\theta}^2 + \dot{\varphi}^2 \operatorname{sen}^2\theta\right) + \frac{\overset{[cm]}{I_z}}{2}\left(\dot{\psi} + \dot{\varphi}\cos\theta\right)^2 - mgl\cos\theta$$

$$\overset{[0]}{L} = \frac{\overset{[0]}{I}}{2}\left(\dot{\theta}^2 + \dot{\varphi}^2 \operatorname{sen}^2\theta\right) + \frac{\overset{[0]}{I_z}}{2}\left(\dot{\psi} + \dot{\varphi}\cos\theta\right)^2 - mgl\cos\theta$$

$$\Rightarrow L = \frac{I}{2}\left(\dot{\theta}^2 + \dot{\varphi}^2 \operatorname{sen}^2\theta\right) + \frac{I_z}{2}\left(\dot{\psi} + \dot{\varphi}\cos\theta\right)^2 - mgl\cos\theta$$

$$\Rightarrow \frac{\partial L}{\partial \dot{\varphi}} = I\dot{\varphi}\operatorname{sen}^2\theta + I_z\left(\dot{\psi} + \dot{\varphi}\cos\theta\right)\cdot\cos\theta = P_\varphi \quad (1)$$

$$\frac{\partial L}{\partial \dot{\psi}} = I_z\left(\dot{\psi} + \dot{\varphi}\cos\theta\right) = P_\psi \quad (2)$$

$$\Rightarrow (1) - (2)\cdot\cos\theta \Rightarrow I\dot{\varphi}\operatorname{sen}^2\theta = P_\varphi - P_\psi\cos\theta$$

$$\Rightarrow \boxed{\dot{\varphi} = \frac{P_\varphi - P_\psi\cos\theta}{I\operatorname{sen}^2\theta}}$$

$$(2) \Rightarrow P_\psi = I_z\left(\dot{\psi} + \dot{\varphi}\cos\theta\right) \Rightarrow \boxed{\dot{\psi} = \frac{P_\psi}{I_z} - \frac{\left(P_\varphi - P_\psi\cos\theta\right)\cos\theta}{I\operatorname{sen}^2\theta}}$$

$$\Rightarrow I\ddot{\theta} = I\dot{\varphi}^2\operatorname{sen}\theta\cos\theta + I_z\left(\dot{\psi} + \dot{\varphi}\cos\theta\right)\cdot -\operatorname{sen}\theta\,\dot{\varphi} + mgl\operatorname{sen}\theta$$

$$= \frac{I\left(P_\varphi - P_\psi\cos\theta\right)^2\operatorname{sen}\theta\cos\theta}{I^2\operatorname{sen}^4\theta} - \frac{P_\psi\cdot\left(P_\varphi - P_\psi\cos\theta\right)\operatorname{sen}\theta}{I\operatorname{sen}^2\theta}$$

$$+ mgl\operatorname{sen}\theta$$

$$\Rightarrow U^{eff} = \frac{\left(P_\varphi - P_\psi\cos\theta\right)^2}{2I\operatorname{sen}^2\theta} + mgl\cos\theta + \frac{P_\psi^2}{2I_z}$$

en equilibrio $\Rightarrow \left.\dfrac{\partial U_{eff}}{\partial \theta}\right|_{\theta^*} = 0$

$$\Rightarrow -\frac{I\,(P_\varphi - P_\psi \cos\theta^*)^2 \,\text{sen}\,\theta^* \cos\theta^*}{I^2 \,\text{sen}^4\theta^*} + \frac{P_\psi \cdot (P_\varphi - P_\psi \cos\theta^*)\cdot \text{sen}\,\theta^*}{I\,\text{sen}^2\theta^*}$$

$$- mgl\,\text{sen}\,\theta^* = 0 \qquad y \quad \theta^* = \frac{\pi}{2}$$

$$\Rightarrow \frac{P_\psi . P_\varphi}{I} - mgl = 0$$

$$\Rightarrow \boxed{P_\psi . P_\varphi = I \cdot mgl}$$

$$\frac{\partial U_{eff}}{\partial \theta} = \frac{P_\psi (P_\varphi - P_\psi \cos\theta)}{I\,\text{sen}\,\theta} - \frac{(P_\varphi - P_\psi \cos\theta)^2 \cos\theta}{I\,\text{sen}^3\theta} - mgl\,\text{sen}\,\theta$$

$$\Rightarrow \frac{\partial^2 U_{eff}}{\partial \theta^2} = \frac{P_\psi^2 \,\text{sen}\,\theta}{I\,\text{sen}\,\theta} - \frac{P_\psi (P_\varphi - P_\psi \cos\theta)\, I \cos\theta}{I^2\,\text{sen}^2\theta}$$

$$- \left[\frac{(2(P_\varphi - P_\psi\cos\theta)\,\text{sen}\,\theta\cos\theta + (P_\varphi - P_\psi\cos\theta)^2 \cdot -\text{sen}\,\theta)}{I\,\text{sen}^3\theta} - \frac{(P_\varphi - P_\psi \cos\theta)^2 \cos\theta \cdot 3 \cdot I\,\text{sen}^2\theta\cos\theta}{I^2\,\text{sen}^6\theta}\right]$$

$$- mgl\,\text{sen}\,\theta$$

$$\Rightarrow \left.\frac{\partial^2 U_{eff}}{\partial \theta^2}\right|_{\theta^*} = \frac{P_\psi^2}{I} + \frac{P_\varphi^2}{I}$$