Ejercicio 03 : Sistemas dinámicos

$$S = \int_{t_1}^{t_2} \left\{ \underbrace{\alpha \dddot{x}\dot{x}}_{\text{derivada total } \frac{d}{dt}\left(\frac{1}{2}\alpha \dot{x}^2\right)} + \frac{m}{2}\dot{x}^2 - \frac{1}{2}kx^2 \right\} dt$$

$$= \int_{t_1}^{t_2} L(\ddot{x}, \dot{x}, x, t)\,dt \rightarrow \text{variando}$$

$$x \longrightarrow x + \delta x \Rightarrow$$

$$S(x+\delta x) - S(x) = \int_{t_1}^{t_2} \left[L(\ddot{x}+\delta\ddot{x}, \dot{x}+\delta\dot{x}, x+\delta x) - L(\ddot{x},\dot{x},x)\right] dt$$

TAYLOR

en nuestro caso 0

$$= S(x+\delta x) - S(x) = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial \ddot{x}}\delta\ddot{x} + \frac{\partial L}{\partial \dot{x}}\delta\dot{x} + \frac{\partial L}{\partial x}\delta x + \cancel{L(\ddot{x},\dot{x},x)} + \cancel{\frac{\partial L}{\partial t}} - \cancel{L(\ddot{x},\dot{x},x,t)} \right] dt$$

$$= \int_{t_1}^{t_2} \left[\underset{②}{\frac{\partial L}{\partial \ddot{x}}\delta\ddot{x}} + \underset{①}{\frac{\partial L}{\partial \dot{x}}\delta\dot{x}} + \frac{\partial L}{\partial x}\delta x \right] dt \text{ pero}$$

① $$\frac{\partial L}{\partial \dot{x}}\delta\dot{x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\delta x\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x$$

② $$\frac{\partial L}{\partial \ddot{x}}\delta\ddot{x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\delta\dot{x}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\right)\delta\dot{x} \qquad \frac{d}{dt}\delta x = \delta\dot{x} = \delta\frac{dx}{dt}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\delta\dot{x}\right) - \left(\frac{d}{dt}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\right)\delta x\right) - \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right)\delta x\right)$$

$$\Rightarrow \delta S(x) = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial x}\delta x + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\delta x\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x + \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\delta\dot{x}\right) - \frac{d}{dt}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\right)\delta x\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right)\delta x \right] dt$$

$$\Rightarrow \delta S = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right) \right] \delta x \, dt$$

$$+ \int_{t_1}^{t_2} \left[\underbrace{\cancel{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\delta x\right)}^{0}}_{\delta x \equiv 0 \text{ en los extremos}} + \underbrace{\cancel{\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\delta \dot{x}\right)}^{0}}_{\delta \dot{x} \equiv 0 \text{ pues no hay variación}} - \underbrace{\cancel{\frac{d}{dt}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\right)\delta x\right)}^{0}}_{\delta x = 0 \text{ en los extremos}} \right] dt$$

$$\Rightarrow \delta S = \int_{t_1}^{t_2} \Big[\underbrace{\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)}_{\text{ecc de Lagrange}} + \underbrace{\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right)}_{\text{termino extra}} \Big] \delta x \, dt \underbrace{= 0}_{\text{condición extremal}}$$

$$\Leftrightarrow \quad \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right) \equiv 0 \quad \Leftrightarrow$$

$$\boxed{\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial x} = 0}$$

$\Rightarrow$ para $L(\ddot{x}, \dot{x}, x) \equiv \alpha \ddot{x}\dot{x} + \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2$ se tiene

1) $\frac{\partial L}{\partial \ddot{x}} = \alpha \dot{x} \Rightarrow \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right) = \frac{d^2}{dt^2}(\alpha \dot{x}) \underbrace{=}_{\alpha = \text{cte}} \boxed{\alpha \dddot{x}}$

2) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}\left(m\dot{x} + \alpha \ddot{x}\right) = m\ddot{x} + \alpha \dddot{x}$

3) $\frac{\partial L}{\partial x} = -kx \Rightarrow$ con esto se tiene la ecc. de mov.

$$\boxed{\cancel{\alpha \dddot{x}} - m\ddot{x} \cancel{- \alpha \dddot{x}} + kx = 0}$$ ecc de movimiento del sistema MAS $\forall \alpha$

Para la interpretación, ~~si $\alpha = 0$~~ $\Rightarrow m\ddot{x} = -kx \Rightarrow$

$\boxed{\ddot{x} = -\omega^2 x}$ $\quad \omega = \sqrt{\frac{k}{m}} \Rightarrow$ resorte Armonico Simple

$\Rightarrow$ No influye en el mov.

~~⇒ α Induce desviación del MAS a borde~~

$$\delta S = \int_{t_1}^{t_2}\Big[\Big(\alpha(\ddot x+\delta\ddot x)(\dot x+\delta\dot x) + \frac{m}{2}(\dot x+\delta\dot x)^2 - \frac{k}{2}(x+\delta x)^2$$

$$-\Big(\alpha\ddot x\dot x + \frac{m}{2}\dot x^2 - \frac{k}{2}x^2\Big)\Big]dt$$

$$= \int_{t_1}^{t_2}\Big[\alpha(\cancel{\ddot x\dot x} + \ddot x\delta\dot x + \delta\ddot x\dot x + \delta\ddot x\delta\dot x) + \frac{m}{2}(\cancel{\dot x^2} + 2\dot x\delta\dot x)$$

$$- \frac{k}{2}(\cancel{x^2} + 2x\delta x) - (\cancel{\alpha\ddot x\dot x} + \cancel{\tfrac{m}{2}\dot x^2} - \cancel{\tfrac{k}{2}x^2}\Big]dt$$

$$= \int_{t_1}^{t_2}\Big[\alpha(\ddot x\delta\dot x + \delta\ddot x\dot x + \delta\ddot x\delta\dot x) + \frac{m}{\cancel{2}}\cancel{2}\dot x\delta\dot x - \frac{k}{\cancel{2}}\cancel{2}x\delta x\Big]dt$$

$$= \int_{t_1}^{t_2}\Big[\alpha(\ddot x\delta\dot x + \delta\ddot x\dot x + \delta\ddot x\delta\dot x) + m\underbrace{\dot x\delta\dot x}_{\frac{d}{dt}\cancel{(\dot x\delta x)}^{0} - \ddot x\delta x} - kx\delta x\Big]dt$$

$$= \int_{t_1}^{t_2}\Big[\alpha(\ddot x\delta\dot x + \delta\ddot x\dot x + \delta\ddot x\delta\dot x) - m\ddot x\delta x - kx\delta x\Big]dt$$

$$= \int_{t_1}^{t_2}\Big[\alpha(\ddot x\delta\dot x + \delta\ddot x\dot x + \underbrace{\delta\dot x\delta\ddot x}_{\frac{1}{2}\frac{d}{dt}\cancel{((\delta\dot x)^2)}^{0}}) - (m\ddot x + kx)\delta x\Big]dt$$

$$= \int_{t_1}^{t_2}\Big[\alpha(\underset{①}{\ddot x\delta\dot x} + \underset{②}{\dot x\delta\ddot x}) - (m\ddot x + kx)\delta x\Big]dt$$

$$\cancel{① \ \ddot x\delta\dot x = \frac{d}{dt}(\ddot x\delta x) - \dddot x\delta x}$$

$$② \ \dot x\delta\ddot x = \frac{d}{dt}(\dot x\delta\dot x) - \ddot x\delta\dot x \Rightarrow \dot x\delta\ddot x + \ddot x\delta\dot x = \frac{d}{dt}(\dot x\delta\dot x)$$

$$\cancel{= \frac{d}{dt}\Big(\frac{d}{dt}(\dot x\delta\dot x) - \ddot x\delta x\Big) - \dot x\delta\dot x}$$

$$\Rightarrow = \int_{t_1}^{t_2}\Big[\frac{d}{dt}\cancel{(\alpha\dot x\delta\dot x)}^{0} - (m\ddot x + kx)\delta x\Big]dt \Rightarrow \boxed{-(m\ddot x + kx) \equiv 0}$$

en los bordes

ecc. de mov. MAS