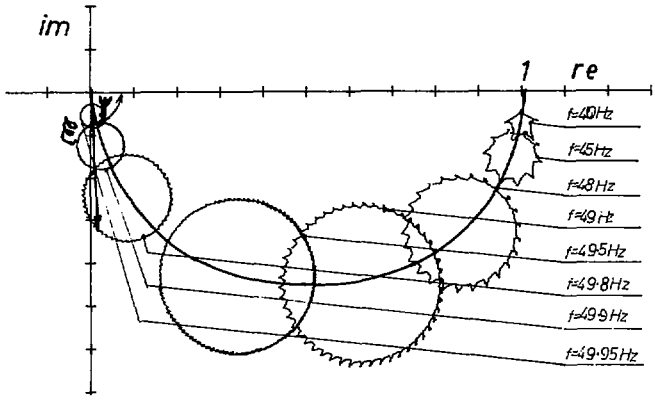

Fig. 8. The family of output phasor trajectories.

By comparison of the notch filter output $\epsilon(t)$ obtained by using (3), (4), and (15), with assumption (11) we get the following expressions:

$$\frac{\mu EC^2}{2(\omega-\omega_0)}\{-2\cos(\omega t+\psi)-k\cos(\omega t-\gamma+2\varphi) - k\cos(\omega t-\gamma+2\varphi-2\delta)\}=\sin\omega t-E\sin(\omega t+\psi), \quad (16)$$

$$\frac{\mu EC^2}{2(\omega-\omega_0)}\{\cos[(\omega-2\omega_0)t+\psi-2\varphi]+\cos[(\omega-2\omega_0)t +\psi-2\varphi+2\delta]+2k\cos[(\omega-2\omega_0)t-\gamma]\} =kE\sin[(\omega-2\omega_0)t-\gamma]. \quad (17)$$

From these we can get four independent equations using amplitude and phase equality. From this set of equations we determine constants E and k and phase angles ψ and γ. It is obvious that (17) for $k=0$ can be valid only if $\delta=\pi/2$, which means that unwanted pulsation disappears for $\delta=\pi/2$ only.

References

[1] B. Widrow, J. M. McCool, M. G. Larimore, and C. R. Johnson, "Stationary and nonstationary learning characteristics of the LMS adaptive filter," *Proc. IEEE*, vol. 64, Aug. 1976.

[2] B. Widrow, J. M. McCool, C. S. Williams, R. H. Hearn, J. R. Zeidler, E. Dong, and R. C. Goodlin, "Adaptive noise cancelling: Principles and applications," *Proc. IEEE*, vol. 63, Dec. 1975.

[3] J. R. Glover, "Adaptive noise canceling applied to sinusoidal interferences," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-25, Dec. 1977.

[4] B. Zupančič, D. Matko, F. Bremšak, P. Sega, R. Karba, "Adaptivni kompenzator motenj (adaptive noise compensator)," Faculty of Elec. Eng., Univ. Ljubljana, Yugoslavia, Tech. Rep., 1977.

A New Error Model for Adaptive Systems

YUAN-HAO LIN AND KUMPATI S. NARENDRA

***Abstract*—A simple error model is described for both discrete and continuous time adaptive systems. The stability of the model for both bounded and unbounded inputs is analyzed for the discrete case and extended to continuous time models. It is shown that $\Delta\phi(k)\in l^2$ in the former case and $\dot{\phi}(t)\in \mathcal{L}^2$ in the latter case due to the presence of a feedback signal and these in turn play an important role in the stability analysis of discrete and continuous time adaptive systems.**

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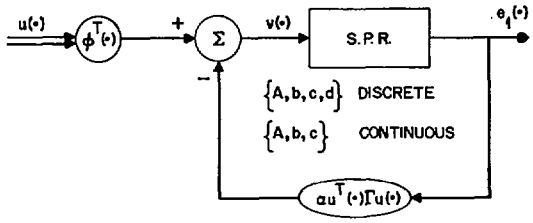

Fig. 1. The new error model.

I. Introduction

Model reference adaptive schemes for the identification and control of continuous and discrete systems have been discussed by numerous authors in recent years [1]–[9]. Among the many approaches that are currently in vogue, the stability approach using Lyapunov theory and the hyperstability theory of Popov have found wide acceptance. The problem of convergence of the parameters of a dynamical system to desired values in identification and control problems is recast, using this approach, to the equivalent asymptotic stability problem of a set of error differential or difference equations.

In a recent paper [9] three such error models were described for continuous systems which arise frequently in problems of adaptation. Discrete versions of these models are also very desirable since in most applications which use a computer as a part of the controller, discrete rather than continuous algorithms are used. While the discrete versions of the first two models can be obtained in a relatively straightforward manner from their continuous counterparts, the third error model has proved to be considerably more difficult. In this paper a simple discrete error model of the third type which permits a complete stability analysis is presented and is also extended to the continuous case. This model has already played a central role in the stability analysis of both discrete and continuous time adaptive control systems [11], [12] and appears to have considerable potential for many other adaptive problems.

II. The New Error Model

Fig. 1 represents the error model of the third prototype which can be used for both discrete and continuous time systems.

In the discrete case the equations describing the model are

$$\begin{aligned} e(k+1)&=Ae(k)+bv(k)\\ e_1(k)&=c^Te(k)+dv(k)\\ v(k)&=\phi^T(k)u(k)-\alpha u^T(k)\Gamma u(k)e_1(k); \qquad \alpha>\frac{1}{2}\ \Gamma=\Gamma^T>0. \end{aligned} \quad (1)$$

In the continuous case the corresponding equations are

$$\begin{aligned} \dot{e}(t)&=Ae(t)+bv(t)\\ e_1(t)&=c^Te(t)\\ v(t)&=\phi^T(t)u(t)-\alpha u^T(t)\Gamma u(t)e_1(t); \qquad \alpha>0\ \Gamma=\Gamma^T>0. \end{aligned} \quad (2)$$

In (1) and (2), $e(\cdot)$ and $u(\cdot)$ are n- and m-dimensional vectors and $v(\cdot)$ and $e_1(\cdot)$ are scalars. A is a matrix, b and c are vectors of suitable dimensions, and d is a positive scalar. The z-transfer function $d+c^T(zI-A)^{-1}b$ in the discrete case and the s-transfer function $c^T(sI-A)^{-1}b$ in the continuous case are strictly positive real.

From Fig. 1 it is seen that the model contains a known linear time-invariant part with a strictly positive real transfer function, a known time-varying gain $\alpha u^T(\cdot)\Gamma u(\cdot)$ which depends on the input signal, in the feedback path and a time varying vector gain $\phi(\cdot)$ in the feedforward path. The vector $\phi(\cdot)$ is assumed to be unknown but changes in $\phi(\cdot)$ ($\Delta\phi$ in the discrete case and $\dot{\phi}$ in the continuous case) can be made using the measured signals $u(\cdot)$ and $e_1(\cdot)$. The objective is to determine suitable adaptive control laws for updating $\phi(\cdot)$ such that in the limit $e_1(k)$ (or $e_1(t)$)$\to 0$ as $k\to\infty$ (or $t\to\infty$). While relatively complex schemes are

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known in the literature [5], our aim is to develop a scheme, particularly for the discrete case, which is comparable in its simplicity to those described in [9]. The principal result of this paper is that the adaptive laws

$$\Delta\phi(k) \triangleq \phi(k+1)-\phi(k) = -\Gamma e_1(k)u(k) \tag{3}$$

and

$$\dot{\phi}(t) = -\Gamma e_1(t)u(t) \tag{4}$$

result in a stable error model in which $e(\cdot)$ and $\phi(\cdot)$ remain uniformly bounded and $\lim_{k\to\infty} e_1(k)$ (or $\lim_{t\to\infty} e_1(t)$)$=0$ if the input $u(\cdot)$ is uniformly bounded. Only the discrete case is considered in detail; the extension of the results to the continuous model is merely outlined.

Lemma: Given an $(n\times n)$ matrix A with all its eigenvalues within the unit circle, a symmetric positive definite matrix Γ, vectors $b,c\in\mathbf{R}^n$ with (A,b) completely controllable and $u(k)$: $Z_+\to\mathbf{R}^m$ whose elements are bounded, the equilibrium states of the set of $n+m$ difference equations (1), (3) is stable and $\lim_{k\to\infty} e(k)=0$ if the transfer function $d+c^T(zI-A)^{-1}b$ is strictly positive real.

Proof: From the discrete version of the Kalman–Yacubovich lemma [10] it is known that if $d+c^T(zI-A)^{-1}b$ is strictly positive real, a matrix $P=P^T>0$ exists such that

$$\begin{aligned} A^TPA-P &= -qq^T-\epsilon L \\ A^TPb &= c/2+\nu q \\ d-b^TPb &= \nu^2 \end{aligned} \tag{5}$$

for some vector q, matrix $L=L^T>0$ and $\epsilon,\nu>0$.

Defining a Lyapunov function candidate for the set of difference equations (1), (3) as

$$V(e(k),\phi(k)) = 2e^T(k)Pe(k)+\phi^T(k)\Gamma^{-1}\phi(k)$$

we obtain

$$\begin{aligned} \Delta V(e(k),\phi(k)) \triangleq \Delta V(k) &= V(k+1)-V(k) \\ &= 2[e^T(k)(A^TPA-P)e(k)+2e^T(k)A^TPbv(k)+b^TPbv^2(k)] \\ &\quad +\phi^T(k+1)\Gamma^{-1}\phi(k+1)-\phi^T(k)\Gamma^{-1}\phi(k). \end{aligned}$$

Choosing the matrix P given by the relations (5):

$$\begin{aligned} \Delta V(k) = -2[e^T(k)q-\nu v(k)]^2-2\epsilon e^T(k)Le(k)+2e_1(k)v(k) \\ +2\phi^T(k)\Gamma^{-1}\Delta\phi(k)+\Delta\phi^T(k)\Gamma^{-1}\Delta\phi(k). \end{aligned}$$

With the adaptive law (3) for $\Delta\phi(k)$ the sum of the last three terms may be expressed as

$$2e_1(k)v(k)-2\phi^T(k)e_1(k)u(k)+e_1^2(k)u^T(k)\Gamma u(k)$$

or

$$(-2\alpha+1)u^T(k)\Gamma u(k)e_1^2(k).$$

Hence

$$\begin{aligned} \Delta V(k) &= -2[e^T(k)q-\nu v(k)]^2-2\epsilon e^T(k)Le(k) \\ &\quad +(-2\alpha+1)u^T(k)\Gamma u(k)e_1^2(k) \\ &< 0 \qquad \text{if } \alpha>\frac{1}{2} \end{aligned} \tag{6}$$

The system (1), (3) is stable and $e(k)$ and $\phi(k)$ are bounded if $e(0)$ and $\phi(0)$ are bounded.

Having established the global stability of the discrete error model we now state some of its other stability properties.

1) The boundedness of $e(k)$ and $\phi(k)$ are assured even when $u(k)$ is not bounded.

2) Furthermore, we can prove $\lim_{k\to\infty} e(k)\to 0$, $\lim_{k\to\infty} e_1(k)\to 0$ whether or not $u(k)$ is bounded. Defining

$$(2\alpha-1)u^T(k)\Gamma u(k) \triangleq \xi(k), \qquad \xi(k)>0$$

we have

$$\Delta V(k) = -2[e^T(k)q-\nu v(k)]^2-2\epsilon e^T(k)Le(k)-\xi(k)e_1^2(k)$$

$$\left|\sum_{k=0}^{\infty}\Delta V(k)\right| = |V(\infty)-V(0)|<\infty \qquad \text{since } \Delta V(k)<0$$

or

$$\sum_{k=0}^{\infty}\left\{2[e^T(k)q-\nu v(k)]^2+2\epsilon e^T(k)Le(k)+\xi(k)e_1^2(k)\right\}<\infty. \tag{7}$$

Hence we conclude $\lim_{k\to\infty} e(k)=0$ whether $u(k)$ is bounded or not. If $\lim_{k\to\infty} u(k)=0$ then by (1) $\lim_{k\to\infty} e_1(k)=0$. If $u(k)$ does not tend to zero as $k\to\infty$ we show that along every subsequence $\{k_j\}$ the error $e_1(k_j)\to 0$ as follows: if along the subsequence $\lim_{k_j\to\infty} u(k_j)=0$ the above result holds and $e_1(k_j)\to 0$.

If $\lim_{k_j\to\infty} u(k_j)\neq 0$ then $\xi(k_j)$ does not tend to zero and hence $e_1(k_j)\to 0$ by (7). Hence whether or not $u(k)$ is bounded, $e_1(k)\to 0$ along every subsequence $\{k_j\}$ and the result follows.

3) From (7) $\lim_{k\to\infty}\xi(k)e_1^2(k)=0$ and since $\Delta\phi(k)=-\Gamma e_1(k)u(k)$ by (3) we conclude that $\Delta\phi(k)\in l^2$ and hence $\lim_{k\to\infty}\Delta\phi(k)=0$. However, we cannot conclude directly that $\phi(k)$ tends to a constant vector ϕ^*.

A similar lemma can also be stated for the continuous time model described by (2) and (4). In such a case, using a Lyapunov function of the form

$$V[e,\phi] = e^TPe+\phi^T\Gamma^{-1}\phi \tag{8}$$

and the Kalman–Yacubovich lemma we obtain that

$$\dot{V}[e,\phi] = -e^T(t)[qq^T+\epsilon L]e(t)-2\alpha e_1^2(t)u^T(t)\Gamma u(t)<0. \tag{9}$$

From (8) and (9) we conclude that $e(t)$ and $\phi(t)$ are uniformly bounded if $e(0)$ and $\phi(0)$ are bounded. Further, from (4) it follows that $\dot{\phi}\in\mathcal{L}^2$. If $u(\cdot)$ and $\dot{u}(\cdot)$ are also assumed to be uniformly bounded from (2) and (9) it follows that $\dot{V}[e,\phi]$ is uniformly bounded and hence $\dot{V}[e,\phi]\to 0$ as $t\to\infty$ or $e(t)\to 0$ as $t\to\infty$ and $\dot{\phi}(t)\to 0$.

In summary, in both the discrete and continuous models the vectors $e(\cdot)$ and $\phi(\cdot)$ are always uniformly bounded. In the discrete case both $e_1(k)$ and $\Delta\phi(k)$ tend to zero as $k\to\infty$ whether or not $u(k)$ is uniformly bounded. In the continuous case for arbitrary $u(t)$ defined for all $t\in\mathbf{R}^+$, we can only conclude that $\dot{\phi}(t)\in\mathcal{L}^2$. If in addition it is known that $u(t)$ and $\dot{u}(t)$ are uniformly bounded then it can be concluded that $\lim_{t\to\infty} e_1(t)=0$ and $\lim_{t\to\infty}\dot{\phi}(t)=0$. In both cases very little can be concluded about the convergence of $\phi(\cdot)$ to a constant vector.

Comments

1) The principal difference between the model described here and those already known [9] lies in the feedback signal $\alpha u^T(\cdot)\Gamma u(\cdot)e_1(\cdot)$. It is the presence of this term which assures that $\Delta\phi(\cdot)\in l^2$ and $\dot{\phi}(\cdot)\in\mathcal{L}^2$. These conditions are essential for the proof of stability of the discrete and continuous adaptive control problems.

2) For continuous time systems the error model can be directly realized by suitably feeding the signal $\alpha u^T(t)\Gamma u(t)e_1(t)$ back to the controller in the adaptive control problem [12]. For discrete systems, while the error model described is convenient for analysis purposes it can be physically realized by using the structure given in Fig. 2.

Conclusion

Three error models have been analyzed in the adaptive literature [9]. The new error model described here is an extension of the third error model in [9] by the use of an additional feedback signal. Both discrete and continuous time versions of the error model are considered here.

For discrete systems, it is shown that $\Delta\phi(k)\in\mathcal{L}^2$ which also implies that $\lim_{k\to\infty}\Delta\phi(k)=0$. In the adaptive problems this corresponds to a slowly varying parameter error vector. Similarly in continuous time systems $\dot{\phi}(t)\in\mathcal{L}^2$. The stability proofs for the adaptive control problem given in [11] and [12] are seen to depend on this behavior of the parameter error vector and this accounts for the importance of the new error model.

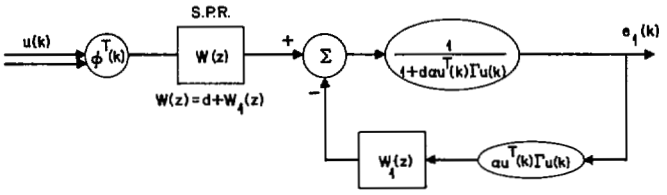

Fig. 2. Physical realization of the new error model for the discrete system.

References

[1] R. L. Carroll and D. P. Lindorff, "An adaptive observer for single-input single-output linear system," *IEEE Trans. Automat. Contr.*, vol. AC-18, pp. 428–435, Oct. 1973.

[2] G. Lüders and K. S. Narendra, "Stable adaptive schemes for state estimation and identification of linear systems," *IEEE Trans. Automat. Contr.*, vol. AC-19, pp. 841–847, Dec. 1974.

[3] K. S. Narendra and P. Kudva, "Stable adaptive schemes for system identification and control—Parts I, II," *IEEE Trans. Syst., Man, Cyber.*, vol. SMC-4, pp. 542–560, Nov. 1974.

[4] P. Kudva and K. S. Narendra, "An identification procedure for discrete multivariable systems," *IEEE Trans. Automat. Contr.*, vol. AC-19, pp. 549–552, Oct. 1974.

[5] I. D. Landau and H. M. Silveira, "A stability theorem with applications to adaptive control," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 305–312, Apr. 1979.

[6] R. V. Monopoli, "Model reference adaptive control with an augmented error signal," *IEEE Trans. Automat. Contr.*, vol. AC-19, pp. 474–484, Oct. 1974.

[7] T. Ionescu and R. V. Monopoli, "Discrete model reference adaptive control with an augmented error signal," *Automatica*, vol. 13, pp. 507–518, Sept. 1977.

[8] K. S. Narendra and L. S. Valavani, "Stable adaptive controller design—Direct control," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 570–583, Aug. 1978.

[9] K. S. Narendra, "Stable identification schemes," in *System Identification: Advances and Case Studies*. New York: Academic, 1976.

[10] C. P. Neumann, "Frequency domain stability criteria in nonlinear automatic control," Doctoral dissertation, Harvard Univ., Cambridge, MA, 1967.

[11] K. S. Narendra and Y.-H. Lin, "Stable discrete adaptive control," this issue, pp. 456–461.

[12] K. S. Narendra, Y.-H. Lin, and L. S. Valavani, "Stable adaptive controller design—Part II: Proof of stability," this issue, pp. 440–448.

A Suboptimal Controller for Minimum Sensitivity of Closed Loop Eigenvalues to Parameter Variations

K. GOMATHI, S. S. PRABHU, AND M. A. PAI

***Abstract*—The design of a suboptimal controller for linear time invariant multivariable systems, which assigns the closed loop eigenvalues at desired locations and minimizes their sensitivity with respect to plant parameters, is outlined. The dominant open loop poles are shifted in groups employing a dyadic structure of state feedback for each group. Freedom available in choosing the constants of proportionality of the different rows of the dyadic matrices is used for eigenvalue sensitivity minimization. Since the final controller matrix is the sum of different dyadic matrices, the sensitivity of the final closed loop poles can be made less than that obtained by shifting all the dominant eigenvalues in one stage using unity rank feedback. An example illustrating the design method is given.**

Introduction

Consider, with the usual notations, a linear time invariant multivariable and controllable system

$$\dot{x}=Ax+Bu \tag{1}$$

where $x\in R^n$ and $u\in R^m$. With state feedback of the form

$$u=Kx \tag{2}$$

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all the eigenvalues of A can be shifted to desired closed loop locations [1], [2]. Freedom available in choosing K, can be utilized to minimize a performance criterion such as eigenvalue sensitivity to variations in parameters [3] of A, gains of controller matrix [4], etc. By restricting to dyadic feedback, the computational complexities inherent in multivariable optimization can be reduced.

Pole assignment in groups [4] and utilization of unity rank feedback to each group retain the computational advantages of using dyadic feedback matrices and result in an improvement in the performance index, at the optimum point. This idea is applied to the pole assignment problem to obtain minimum closed loop pole sensitivity to parameter variations.

Problem

For a dyadic feedback

$$u=qk^Tx \tag{3}$$

the system given by (1) can be represented as

$$\dot{x}=Ax+\hat{b}\hat{u} \tag{4}$$

where $\hat{b}(\triangleq Bq)\in R^n$ and $\hat{u}(\triangleq k^Tx)$ is a scalar.

The corresponding closed loop system is given by

$$\dot{x}=A_cx \tag{5}$$

where $A_c=A+Bqk^T$.

The sensitivity S_{jl}^i of the ith closed loop eigenvalue λ_i to variations in the jlth parameter of A is given by [3]

$$S_{jl}^i=\frac{1}{g'(\lambda_i)}\operatorname{tr}\left[P(q)\operatorname{adj}(\lambda_iI-A_{\text{ph}})P^{-1}(q)\frac{\partial A}{\partial a_{jl}}\right] \tag{6}$$

where a_{jl} is the jlth element of A, $g(\lambda)$ denotes the characteristic polynomial of A_c, $g'(\lambda_i)=\partial g/\partial\lambda|_{\lambda=\lambda_i}$. $P(q)$ represents the transformation matrix which transforms the single input system (4) to its phase variable canonical form and A_{ph} is the companion form of A_c. For given closed loop eigenvalue locations S_{jl}^i is a function of q and q can be determined to minimize a sensitivity based performance index J of the general form

$$J=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\left(S_{jl}^i\right)^2. \tag{7}$$

k can then be determined to do the desired pole assignment [1], [2]. $P(q)$ is expressible as a function of q using the Leverrier algorithm [5].

Let the dominant eigenvalues ($<n$) be shifted in "r" groups. Let

$$u=\left(\sum_{i=1}^{r}q^{(i)}k^{(i)^T}\right)x. \tag{8}$$

The system can be represented by

$$\dot{x}=Ax+\sum_{i=1}^{r}\tilde{b}_i\tilde{u}_i \tag{9}$$

where

$$\tilde{b}_i\triangleq Bq^{(i)} \tag{10}$$

and

$$\tilde{u}_i\triangleq k^{(i)^T}x. \tag{11}$$

Assignment of the "r" groups of dominant eigenvalues with sensitivity minimization is done as follows.

Firstly $q^{(1)}$ is determined to minimize J (7) with respect to the closed loop eigenvalues of group 1 and the eigenvalues which are nondominant. Then $k^{(1)}$ is determined for the pole assignment in group 1 [1], [2]. The

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