XI. Conclusion

The paper has analyzed a general class of discrete-time adaptive control algorithms and has shown that, under suitable conditions, they will be globally convergent. The algorithms have a very simple structure and are applicable to both single-input single-output and multiple-input multiple-output systems with arbitrary time delays provided only that a stable control law exists to achieve zero tracking error. The results resolve a long standing question in adaptive control regarding the existence of simple, globally convergent adaptive algorithms.

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Stable Discrete Adaptive Control

KUMPATI S. NARENDRA, FELLOW, IEEE, AND YUAN-HAO LIN

***Abstract*—The paper presents a proof of stability of the model reference adaptive control problem for the discrete case.**

I. Introduction

At present there is widespread interest in the stable adaptive control of unknown linear time-invariant plants using input-output data. Schemes have been suggested for both direct [1]–[3] and indirect [4], [5] control of continuous as well as discrete [6], [7] systems and the equivalence of the two schemes in some cases has also been demonstrated [4], [5]. Probably the single most important problem to arise in the course of these investigations concerns the proof of stability of the overall adaptive control loop.

Monopoli [1] proposed a scheme for continuous systems involving an auxiliary signal fed into the reference model and a corresponding augmented error between model and plant outputs. Narendra and Valavani [2], using positive real operators, suggested a similar approach and clarified the resulting stability problem when the relative degree of the plant is greater than or equal to three. They offered a conjecture that the adaptive loop would also be stable for the general case. Feuer and Morse [3] proposed a stable solution to the adaptive control problem but the resulting controller is much too complex for use in practical applications. Thus, the search has continued for a controller with a simple structure which will assure the global asymptotic stability of the adaptive loop. The results presented in this paper demonstrate the desired stability behavior for discrete versions of the simple controllers suggested in [1] and [2]. Similar results have also been reported recently in [9] and [10] for the discrete adaptive control problem and in [11] and [12] for the continuous case.

This paper examines the discrete version of the problem considered in [2] recapitulating the basic philosophy as well as the specific technique used for the design of the adaptive controller in that paper. Hence the first few sections of this paper have been considerably condensed and the interested reader is referred to the earlier work for all details. The principal contribution made here is the verification of the conjecture made in [2] regarding the stability of the adaptive loop, for the discrete problem when an additional feedback signal suggested in [8] is used. Accordingly most of the paper is devoted to the proof of stability. While the proof given in [11] for continuous systems can be directly extended to the discrete case, we present here a simpler proof which is valid for discrete systems.

II. Statement of the Problem

A single-input single-output discrete linear time-invariant plant P is described by the input–output pair $\{u(k), y_p(k)\}$ and can be represented by the transfer function

$$W_p(z) = k_p \frac{Z_p(z)}{R_p(z)} \tag{1}$$

where $W_p(z)$ is proper, with $R_p(z)$ a monic polynomial of degree n, $Z_p(z)$ a monic stable[1] polynomial of degree $m<n$, and k_p a constant gain parameter. The integer $n-m$ is called the relative degree of the plant. We assume that only m, n and the sign of k_p as well as an upper bound on $|k_p|$ are known, while the coefficients of Z_p and R_p are unknown.

A reference model M whose output $y_M(k)$ represents the behavior desired from the plant when augmented by a suitable controller can be represented by the transfer function

$$W_M(z) \triangleq k_M \frac{Z_M(z)}{R_M(z)} \tag{2}$$

where $R_M(z)$ and $Z_M(z)$ are monic stable polynomials of degrees n and $r<m$ respectively and k_M is a constant. Hence the relative degree of the model is assumed to be greater than or equal to that of the plant. The reference input $r(k)$ to the model is specified and is assumed to be uniformly bounded.

The adaptive control problem is to determine a suitable control function $u(k)$ such that

$$y_p(k) - y_M(k) \to 0 \qquad \text{as } k \to \infty. \tag{3}$$

For the sake of simplicity we shall assume that $r=m$. As in the continuous case, the solution to the above problem may be divided into two parts. The first part which is algebraic in nature addresses itself to the realizability of a suitable controller structure. It can be shown exactly as in the continuous case [2] that a controller can be found which can achieve (3) with a fixed set of parameters. In the following section the

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[1] With all zeros inside the unit circle.

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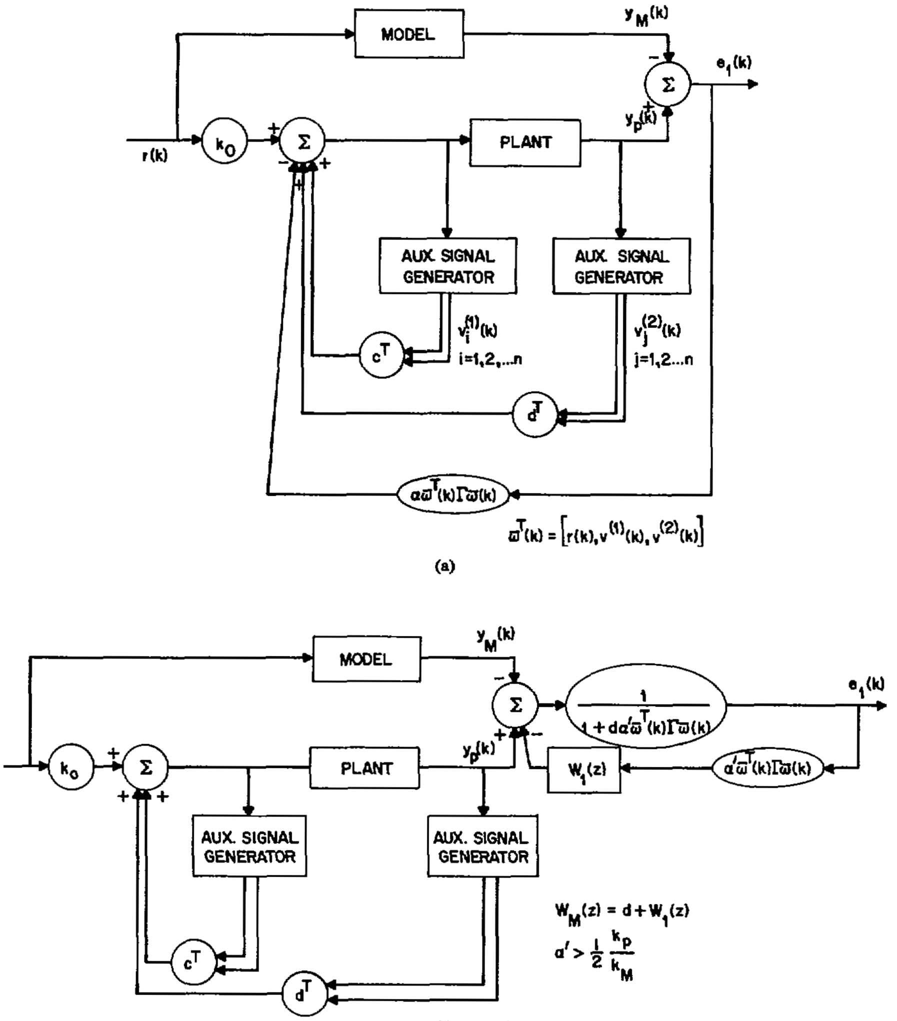

Fig. 1. (a) Adaptive controller for $m=n$. (b) Physical realization of (a).

equations describing the controller are merely stated. The second part is analytic in nature and deals with the stability of the adaptive error equations. Again, it is found that when $m=n$ (or $m=n-1$ in the continuous case) the adaptive equations can be shown relatively easily to be globally asymptotically stable. Hence our main interest is in the case $m<n-1$ and auxiliary inputs have to be fed into the reference model. The statement of the stability problem and the proof of stability are considered in detail in Section V.

III. Structure of the Adaptive Controller

As in [2], two different structures are needed for the discrete adaptive control problem corresponding to the two cases $m=n$ and $m<n-1$. When $m=n$ and the model transfer function is assumed to be positive real,[2] the simple structure shown in Fig. 1(a) can be used.[3] For the case when $m<n-1$, as described in [2], an auxiliary signal has to be fed into the model and the corresponding structure is shown in Fig. 2. (The physical realization of this can be achieved along the same lines as in Fig. 1(b).) A brief description of the controller structure for the two cases is given below.

Case i) ($m=n$): The controller consists of $(2n+1)$ adjustable parameters which are the elements of a parameter vector $\bar{\theta}(k)$ defined by

$$\bar{\theta}^T(k) \triangleq [k_0(k), c_1(k), \cdots c_n(k), d_1(k), d_2(k), \cdots d_n(k)].$$

[2]There is no loss of generality here since by prefiltering the model can be made positive real.

[3]Since the plant in this case has a direct transfer, Fig. 1(a) involves an algebraic loop and is used only for purposes of analysis. The physical realization of the loop is shown in Fig. 1(b).

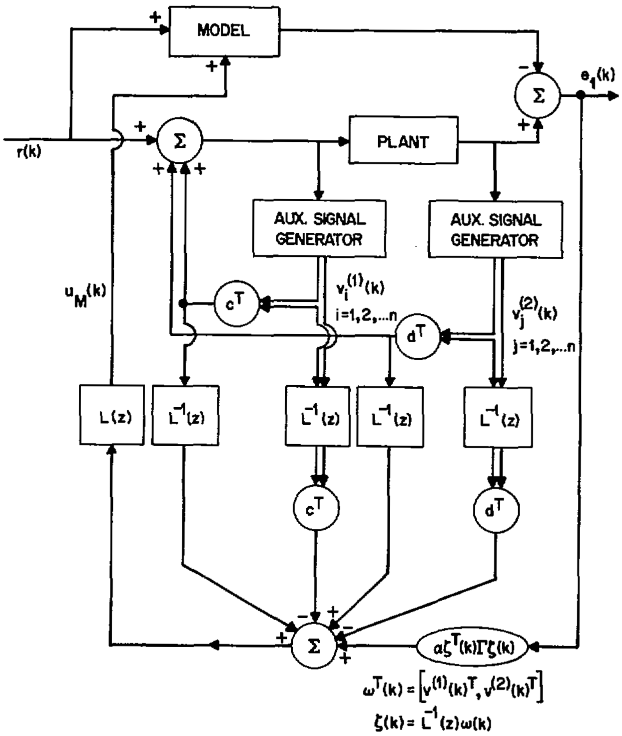

Fig. 2. Adaptive controller for $m<n-1$.

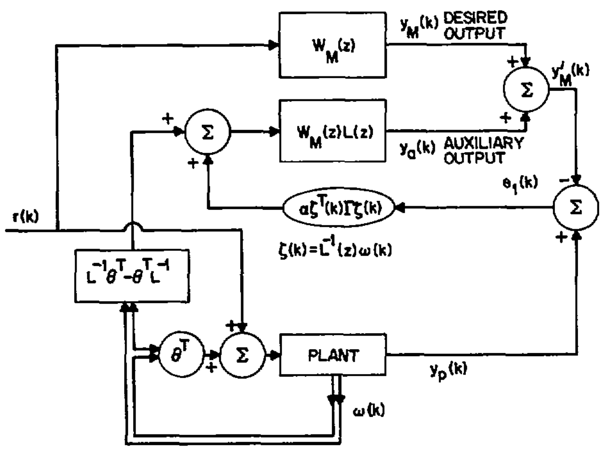

Fig. 3. Equivalent representation of Fig. 2.

Two identical auxiliary signal generators of dimension "n" having state variables $v^{(1)}(k)$ and $v^{(2)}(k)$ and inputs $u(k)$ and $y_p(k)$, respectively, as shown in Fig. 1(a), from part of the controller. If a vector $\bar{\omega}(k)$ is defined as

$$\bar{\omega}(k)^T=[r(k),v^{(1)}(k)^T,v^{(2)}(k)^T],$$

the signal fedback into the plant may be represented by

$$\bar{\theta}^T(k)\bar{\omega}(k)-\alpha\bar{\omega}(k)^T\Gamma\bar{\omega}(k)e_1(k). \tag{4}$$

The first term in (4) represents a linear combination of the elements of $\bar{\omega}(k)$ and corresponds to the feedback signal in the continuous case. The second term which depends on the output error $e_1(k)$ is found essential to establish the stability of the error equations as described in [8]. If $\bar{\omega}(k)$ is bounded as the adaptation proceeds, this term is seen to tend to zero with $e_1(k)$.

Following the results in [2], it can be shown that a constant vector $\bar{\theta}^*$ exists such that when $\bar{\theta}(k)\equiv\bar{\theta}^*$ the transfer function of the plant together with the controller matches that of the reference model.

The adaptive control problem is to determine the law for updating $\bar{\theta}(k)$ such that $\bar{\theta}(k)\to\bar{\theta}^*$ as $k\to\infty$ while maintaining overall system stability.

Case ii) ($m<n-1$): With no loss of generality,[4] we can assume that $L(z)$, a rational function in z (with $L^{-1}(z)$ a strictly proper minimum phase function), exists such that $W_M(z)L(z)$ is strictly positive real. However, since $L(z)$ is not physically realizable, the same modification as that suggested in the continuous case has to be used here as well. As shown below, this involves feeding back signals into both plant and model such that the error equations have the same form as in Case i) (see Section IV). Fig. 3 indicates the structure of the controller when the plant gain k_p is known; for simplicity, it is assumed that $k_p=k_M=1$.[5] Since in this case only $2n$ parameters have to be ajdusted, we define $\theta^T(k)=[c_1(k),c_2(k),\cdots,c_n(k),d_1(k),\cdots,d_n(k)]$ and $\omega^T(k)=[v^{(1)}(k)^T,v^{(2)}(k)^T]$.

The signal fed back into the plant is

$$u_p(k)=\theta^T(k)\omega(k),$$

while the signal fed back into the model is

$$u_M(k)=L(z)[\{L^{-1}(z)\theta^T(k)-\theta^T(k)L^{-1}(z)\}\omega(k)$$
$$+\alpha\zeta^T(k)\Gamma\zeta(k)e_1(k)] \qquad (\text{where} L^{-1}(z)\omega(k)=\zeta(k))$$

so that the resulting error equations have the form required to generate stable adaptive laws as described in Section IV.

[4]It is obvious that a rational function $L(z)$ with denominator polynomial $Z_M(z)$ exists such that $W_M(z)L(z)$ is strictly positive real. A particular choice of $L(z)$ is $W_M^{-1}(z)$ and results in considerable simplification of the analysis.

[5]If $k_M\neq k_p$, an additional adjustable parameter has to be used to generate the auxiliary signal $u_M(k)$ described later in this section. The analysis presented here also carries over to this more general case.

While $L(z)$ is not physically realizable, $W_M(z)L(z)$ can be realized and hence the overall system is as shown in the Fig. 3.

A Special Case: As mentioned earlier, considerable simplification is achieved by choosing $L^{-1}(z)=W_M(z)$. Further, if $W_M(z)$ is chosen to be equal to z^{-d} where d is a positive integer, the problem is to follow the reference input delayed by d steps [i.e., $r(k-d)$] by the output of the plant.

From Figs. 2 and 3 we have

$$y_m(k)=r(k-d); \qquad \zeta(k)=\omega(k-d);$$
$$y_a(k)=[\theta^T(k-d)\omega(k-d)-\theta^T(k)\omega(k-d)]$$

and

$$e_1(k)=\frac{y_p(k)-r(k-d)-y_a(k)}{1+\alpha\omega^T(k-d)\Gamma\omega(k-d)}.$$

The adaptive laws are given by

$$\Delta\theta(k)=-\Gamma e_1(k)\omega(k-d).$$

If tapped delay lines are used as the auxiliary signal generators in Fig. 2 to generate $\omega(k)$, the results are similar to those obtained in [9].

IV. The Error Equations

Let the parameter vector $\bar{\theta}(k)$ be expressed as[6]

$$\bar{\theta}(k)=\bar{\theta}^*+\bar{\phi}(k)$$

where $\bar{\phi}(k)$ represents the parameter error vector at time k.

Case i) ($m=n$): The output error $e_1(k)$ in Fig. 1 may be expressed as

$$e_1(k)=\frac{k_p}{k_M}[W_M(z)]\{\bar{\phi}^T(k)\bar{\omega}(k)-\alpha\bar{\omega}^T(k)\Gamma\bar{\omega}(k)e_1(k)\}$$

where $W_M(z)$ is a strictly positive real operator. From the recent results in [8] (also given in detail in the next section), it is seen that if $\bar{\phi}(k)$ is updated according to the law

$$\bar{\phi}(k+1)=\bar{\phi}(k)-\Gamma e_1(k)\bar{\omega}(k)$$

$e_1(k)\to0$ whether or not $\bar{\omega}(k)$ is bounded. Since in this case the (desired) output $y_m(k)$ of the model is uniformly bounded, the plant output will also be uniformly bounded and approach the desired output asymptotically.

Case ii) ($m<n-1$): The augmented error $e_1(k)$ in Figs. 2 and 3 satisfies the error equation

$$e_1(k)=[W_M(z)L(z)]\{\phi^T(k)\zeta(k)-\alpha\zeta^T(k)\Gamma\zeta(k)e_1(k)\}$$

where $W_M(z)L(z)$ is a strictly positive real transfer function and as defined earlier

$$L^{-1}(z)\omega(k)=\zeta(k).$$

By the same arguments as in Case i) it follows easily that if the adaptive law

$$\phi(k+1)=\phi(k)-\Gamma e_1(k)\zeta(k)$$

is used, then

$$e_1(k)\to0 \qquad \text{as } k\to\infty.$$

However, it no longer follows that the plant output will be bounded since the model output (which is due to both reference and the auxiliary inputs) may be unbounded. Hence, to prove the global asymptotic stability of the adpative system it is necessary to show that neither the plant output nor the model output can be unbounded, in other words, verify for the discrete case the conjecture made in [2] for continuous systems. The rest of this paper is devoted entirely to this problem

[6]This applies to Case i). For Case ii), $\theta(k)=\theta^*+\phi(k)$.

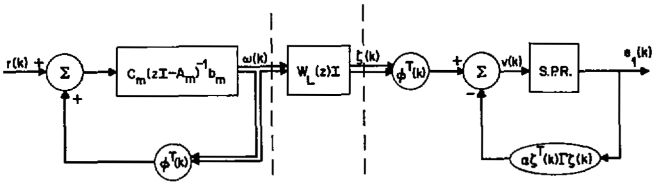

Fig. 4. Three parts of adaptive controller for analysis of stability.

V. Verification of the Conjecture by Narendra and Valavani for Discrete Systems

A. Description of the Error Model

The error model whose stability is to be analyzed is a complex vector nonlinear difference equation. For convenience of analysis we shall consider it in three separate parts which correspond to the three parts of the system shown in Fig. 4.

Part I—The Plant Feedback Loop: The plant together with the controller can be described by the vector difference equation

$$x(k+1)=A_m x(k)+b_m[\phi^T(k)\omega(k)+r(k)]$$
$$\omega(k)=C_m x(k). \tag{5}$$

$\omega(k)$ represents the output vector of interest, $x(k)$ is the state vector of the plant together with the controller, and when $\phi(k)$, the parameter error vector, is identically zero, the plant and model transfer functions match exactly. The reference input $r(k)$ is uniformly bounded and the matrices A_m and C_m, and vector b_m are of appropriate dimensions. As described in the previous section, A_m is a $(3n\times 3n)$ stable matrix, b_m is a $(3n\times 1)$ vector, and C_m is a $(2n\times 3n)$ matrix.

Part II—The Prefilter: The second part of the system shown in Fig. 4 consists of a diagonal transfer matrix all of whose elements are the same and equal to $W_L(z)=L^{-1}(z)$ or

$$\zeta_i(k)=L^{-1}(z)\omega_i(k). \tag{6}$$

$L^{-1}(z)$ is assumed to be an asymptotically stable system of relative degree $n-m$, as described earlier (number of poles − number of zeros = $n-m$), and of minimum phase.

Part III—The Error Model [8]*:* The third part is the model of the error equations described in the previous section and consists of a strictly positive real transfer function in the feedforward path and feedforward and feedback gains $\phi^T(k)$ and $\alpha\zeta^T(k)\Gamma\zeta(k)$, respectively, as shown in Fig. 4. It can also be represented by a $3n$ order difference equation

$$e(k+1)=A_m e(k)+bv(k)$$
$$e_1(k)=c^T e(k)+dv(k) \tag{7}$$
$$v(k)=\phi^T(k)\zeta(k)-\alpha\zeta^T(k)\Gamma\zeta(k)e_1(k)$$
$$\alpha>\tfrac{1}{2}, \Gamma=\Gamma^T>0$$

where $d+c^T(zI-A_m)^{-1}b$ is strictly positive real.

The parameter error vector $\phi(k)$ [and hence the parameter vector $\theta(k)$] is adjusted according to the law

$$\phi(k+1)=\phi(k)-\Gamma e_1(k)\zeta(k)$$
$$(\text{or}\quad \theta(k+1)=\theta(k)-\Gamma e_1(k)\zeta(k)). \tag{8}$$

The four sets of difference equations (5), (6), (7), and (8) completely determine the error model of the overall discrete system. The signal $\zeta(k)$ and output error $e_1(k)$ determine how $\phi(k)$ is updated but this, in turn, determines the nature of $\omega(k)$ and $\zeta(k)$.

B. Statement of the Conjecture

The conjecture in [2] when applied to the problem above may be stated as follows.

If $\phi(k)$ is adjusted according to the law (8) to keep the error $e_1(k)$ bounded, the outputs of the plant, i.e., $\omega(k)$ will also be bounded.

Equivalently, the conjecture implies that the overall nonlinear system described by (5), (6), (7), and (8) is globally stable and that all the signals are uniformly bounded.

Since the stability in the large of the above nonlinear system is intractable, we shall consider the three linear blocks in Fig. 4 separately to simplify the analysis.

C. A Qualitative Analysis

Part I of Fig. 4 is a feedback loop with a stable time-invariant forward path and a time-varying gain vector $\phi(k)$ in the feedback path. The output vector $\omega(k)$ can be either uniformly bounded or unbounded. In the former case $\zeta(k)$, the output of the prefilter is also uniformly bounded and the behavior of the error model and, hence, that of the entire system are completely known. If, however, it is assumed that $\omega(k)$ and, hence, $\zeta(k)$ are unbounded, the analysis in the following section shows that we are led to a contradiction. Hence, only the first alternative is possible (i.e., $\omega(k)$ is uniformly bounded) and the conjecture is verified.

Before proceeding to give an analytic proof, we present here a brief qualitative analysis of the various steps involved.

It is first shown in Section V-D, using the notation defined in the Appendix, that the input to the error model $\phi^T(k)\zeta(k)$ satisfies the relation

$$\phi^T(k)\zeta(k)=o[\|\zeta(k)\|]$$

and

$$\Delta\phi(k)\to 0 \text{ as } k\to\infty.$$

Since $W_L(z)$ is asymptotically stable and minimum phase, it is shown in the Appendix that

$$\sup_{k>\nu}\|\omega(\nu)\|\sim\sup_{k>\nu}\|\zeta(\nu)\|$$

and from Lemma 1 in the Appendix, it is concluded that

$$\phi^T(k)\omega(k)=o\Big[\sup_{k>\nu}\|\omega(\nu)\|\Big]. \tag{9}$$

Since $r(k)$ is uniformly bounded and $\phi^T(k)\omega(k)$ is the feedback signal in the plant feedback loop, it is concluded from (9) that $\omega(\cdot)$ and hence $\zeta(\cdot)$ must also be uniformly bounded.

D. Proof of the Conjecture

The Error Model [8]*:* From the discrete version of the Kalman–Yacubovich lemma if $d+c^T(zI-A_m)^{-1}b$ is strictly positive real, a matrix $P=P^T>0$ and a vector q exist such that

$$A_m^T PA_m-P=-qq^T-\epsilon L;\qquad A_M^T Pb=c/2+\nu q;\qquad d-b^T Pb=\nu^2 \tag{10}$$

for some $L=L^T>0$ and scalars $\epsilon,\nu>0$.

If a Lyapunov function candidate for the set of equations (10) is chosen as

$$V[e(k),\phi(k)]\triangleq V(k)=2e(k)^T Pe(k)+\phi^T(k)\Gamma^{-1}\phi(k), \tag{11}$$

it was shown in [8] that $\Delta V(e(k),\phi(k))\triangleq\Delta V(k)=V(k+1)-V(k)$ may be expressed as

$$\Delta V(k)=-2[e^T(k)q-\nu v(k)]^2-2\epsilon e^T(k)Le(k)$$
$$+(1-2\alpha)\zeta^T(k)\Gamma\zeta(k)e_1^2(k)$$
$$\leqslant 0\qquad \text{if } \alpha>\frac{1}{2}. \tag{12}$$

Hence the system is stable and $e(k)$ and $\phi(k)$ are bounded if $e(0)$ and $\phi(0)$ are bounded. Furthermore, from (11) and (12) it follows that

$$e(k)\to 0\quad \zeta^T(k)\Gamma\zeta(k)e_1^2(k)\to 0\qquad \text{and } e_1(k)\to 0\qquad \text{as } k\to\infty \tag{13}$$

whether or not $\zeta(k)$ is uniformly bounded. Since $\Delta\phi(k)=-\Gamma e_1(k)\zeta(k)$, from (13) we have

$$\Delta\phi(k)\to 0 \qquad \text{as } k\to\infty. \tag{14}$$

Further since $e_1(k)=c^Te(k)+dv(k)$, the input $v(k)$ to the strictly positive real transfer function also tends to zero. Hence we have

$$\phi^T(k)\zeta(k)-\alpha\zeta^T(k)\Gamma\zeta(k)e_1(k)\to 0 \qquad \text{as } k\to\infty. \tag{15}$$

Since $\alpha\zeta^T(k)\Gamma\zeta(k)e_1(k)=-\alpha\zeta^T(k)\Delta\phi(k)$ by (14) and (15),

$$\phi^T(k)\zeta(k)=o[\|\zeta(k)\|]. \tag{16}$$

The Prefilter: By (6), we have

$$W_L(z)\omega(k)=\zeta(k).$$

Since the transfer function $W_L(z)$ is asymptotically stable and of minimum phase, we have from Lemma 1 in the Appendix, using (14) and (16),

$$\phi^T(k)\omega(k)=o\Big[\sup_{k>\nu}\|\omega(\nu)\|\Big]. \tag{17}$$

The Plant Feedback Loop: The feedback loop in Part I is described by the difference equation

$$\begin{aligned}x(k+1)&=[A_m+b_m\phi^T(k)C_m]x(k)+b_mr(k)\\&=A_mx(k)+b_m\phi^T(k)\omega(k)+b_mr(k).\end{aligned} \tag{18}$$

Since $r(k)$ is uniformly bounded and A_m is an asymptotically stable matrix it follows that if $\omega(k)$ is unbounded

$$\|x(k)\|<c_1\sup_{k>\nu}|\phi^T(\nu)\omega(\nu)|+c_2 \qquad c_1,c_2>0. \tag{19}$$

By (17), $\phi^T(k)\omega(k)=o[\sup_{k>\nu}\|\omega(\nu)\|]$ and hence

$$\begin{aligned}\|x(k)\|&=o\Big[\sup_{k>\nu}\|C_mx(\nu)\|\Big]\\&<o\Big[\sup_{k>\nu}\|x(\nu)\|\Big]\end{aligned} \tag{20}$$

which is a contradiction if $x(k)$ is unbounded. Hence $x(k)$, $\omega(k)$, and $\zeta(k)$ are uniformly bounded and the adaptive control system is stable. Further, the auxiliary input to the model is

$$u_M(k)=L(z)\Big\{\big[L^{-1}(z)\phi(k)-\phi(k)L^{-1}(z)\big]^T\omega(k)+\alpha e_1(k)\zeta^T(k)\Gamma\zeta(k)\Big\}$$

and since $\phi^T(k)\omega(k)$, $\phi^T(k)\zeta(k)$, and $e_1(k)\to 0$ as $k\to\infty$, $u_M(k)\to 0$ as $k\to\infty$, and the plant output asymptotically approaches the desired output.

VI. Conclusion

The paper presents a proof of the stability of the adaptive control system suggested by Narendra and Valavani [2] for the discrete case with an additional feedback signal. The same proof could also be used to show that the discrete controller suggested by Ionescu and Monopoli [6] is also stable.

Appendix

Definition 1: Let $\{x(k)\}$ and $\{y(k)\}$ be two sequences such that $x(\cdot)$ and $y(\cdot)\in l_e^\infty$. If there exists a sequence $\{\beta(k)\}$, with $\beta(k)\to 0$ as $k\to\infty$ such that $y(k)=\beta(k)x(k)$ then we denote

$$y(k)=o[x(k)].$$

Definition 2: Let $\{x(k)\}$ and $\{y(k)\}$ be two sequences. If there exists a positive constant M such that

$$|y(k)|<M|x(k)| \qquad \text{for all } k\in N$$

we denote it by

$$y(k)=0[x(k)].$$

Definition 3: If $\{x(k)\}$ and $\{y(k)\}$ are two sequences such that $x(k)=0[y(k)]$ and $y(k)=0[x(k)]$, we say that the two sequences are equivalent and denote it by

$$x(k)\sim y(k).$$

The following definition is found to be useful while describing two sequences which evolve at the same rate but are not equivalent.

Definition 4: Two sequences $\{x(k)\}$ and $\{y(k)\}$ are said to grow at the same rate if

$$\sup_{k>\nu}|x(\nu)|\sim\sup_{k>\nu}|y(\nu)|.$$

It follows that two sequences which are equivalent grow at the same rate but not vice versa.

Let $W_L(z)$ be a rational transfer function of a linear time-invariant discrete system with all its poles and zeros within the unit circle and input and output $x(\cdot)$ and $y(\cdot)$, respectively. Let $h(k)$ be the impulse response of $W_L(z)$. Since $h\in l_1$, it follows that

$$|y(k)|<c_1\sup_{k>\nu}|x(\nu)|+c_2 \tag{A.1}$$

where c_1 and c_2 are positive constants. Hence

$$\sup_{k>\nu}|y(\nu)|=0\Big[\sup_{k>\nu}|x(\nu)|\Big].$$

Considering $y(k)$ as the input and $x(k)$ as the output, we also have [since $W_L^{-1}(z)$ is stable]

$$|x(k)|<c_1'\sup_{k+r_1>\nu}|y(\nu)|+c_2' \tag{A.2}$$

where c_1' and c_2' are positive constants and r_1 is the relative degree (i.e., number of poles − number of zeros) of $W_L(z)$. If the rate at which the sequence $y(k)$ can grow is bounded (e.g., any linear system with bounded coefficients can have only solutions which grow geometrically), it follows from (A.2) that

$$|x(k)|<c_1''\sup_{k>\nu}|y(\nu)|+c_2'. \tag{A.3}$$

c_1'' is a positive constant and $c_1''=c_1'|\lambda_1|^{r_1}$ where λ_1 denotes the maximum rate at which $y(k)$ can grow.

In view of (A.1) and (A.3)

$$\sup_{k>\nu}|x(\nu)|\sim\sup_{k>\nu}|y(\nu)| \tag{A.4}$$

and $\{x(k)\}$ and $\{y(k)\}$ grow at the same rate.

Lemma 1: Let $W_L(z)$ be a transfer matrix with input vector $\omega(k)$ and output vector $\zeta(k)$ where $W_L(z)$ is an asymptotically stable and minimum phase transfer function and Part I is the unit matrix. $\omega(k)$ and $\zeta(k)$ are assumed to grow at most geometrically. Further, suppose that there is a vector $\phi(k)$ such that

$$\phi^T(k)\zeta(k)=o[\|\zeta(k)\|] \tag{A.5}$$

and

$$\Delta\phi(k)\to 0 \qquad \text{as } k\to\infty. \tag{A.6}$$

Then

$$\phi^T(k)\omega(k)=o\Big[\sup_{k>\nu}\|\omega(\nu)\|\Big]. \tag{A.7}$$

Proof: It follows from (A.4) that for each pair of components of $\omega(\cdot)$ and $\zeta(\cdot)$, we have

$$\sup_{k>\nu}|\omega_i(\nu)|\sim\sup_{k>\nu}|\zeta_i(\nu)| \tag{A.8}$$

or

$$\sup_{k>\nu} \|\omega(\nu)\| \sim \sup_{k>\nu} \|\zeta(\nu)\|. \quad \text{(A.9)}$$

Let

$$W_L(z) = \frac{B(z)}{A(z)} = \frac{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_0}. \quad \text{(A.10)}$$

Then we have the vector equation

$$A(z)\zeta(k) = B(z)\omega(k). \quad \text{(A.11)}$$

Premultiply both sides of (A.11) by $\phi^T(k)$; we obtain

$$\phi^T(k)A(z)\zeta(k) = \phi^T(k)B(z)\omega(k). \quad \text{(A.12)}$$

Since

$$\begin{aligned}\phi^T(k)A(z)\zeta(k) &= \phi^T(k)[\zeta(k+n) + a_{n-1}\zeta(k+n-1) + \cdots + a_0\zeta(k)] \\ &= \phi^T(k+n)\zeta(k+n) + a_{n-1}\phi^T(k+n-1)\zeta(k+n-1) \\ &\quad + \cdots + a_0\phi^T(k)\zeta(k) \\ &\quad + [\phi^T(k) - \phi^T(k+n)]\zeta(k+n) + a_{n-1} \\ &\quad \cdot [\phi^T(k) - \phi^T(k+n-1)]\zeta(k+n-1) \ldots \\ &\quad + a_1[\phi^T(k) - \phi^T(k+1)]\zeta(k+1) \end{aligned} \quad \text{(A.13)}$$

by (A.5), (A.6), (A.9), and the fact that $\zeta(k)$ can grow at most geometrically, (A.13) implies that

$$\phi^T(k)A(z)\zeta(k) = o\Big[\sup_{k>\nu} \|\omega(\nu)\|\Big]. \quad \text{(A.14)}$$

Similarly,

$$\phi^T(k)B(z)\omega(k) = B(z)\phi^T(k)\omega(k) + o\Big[\sup_{k>\nu} \|\omega(\nu)\|\Big]. \quad \text{(A.15)}$$

Therefore, by (A.12), (A.14), and (A.15), it follows that

$$B(z)\phi^T(k)\omega(k) = o\Big[\sup_{k>\nu} \|\omega(\nu)\|\Big]. \quad \text{(A.16)}$$

Since $B(z)$ has all its zeros inside the unit circle,

$$\phi^T(k)\omega(k) = 0\Big\{o\Big[\sup_{k>\nu} \|\omega(\nu)\|\Big]\Big\} \quad \text{(A.17)}$$

and the lemma is proved.

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Microprocessor Requirements for Implementing Modern Control Logic

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***Abstract*—Analytical procedures for establishing microprocessor accuracy, computational capability, and memory requirements for implementing linear-quadratic-Gaussian (LQG) control logic were developed and evaluated. The developed procedures were evaluated and illustrated by application to a linearized fifth-order F100 turbofan engine model. Results were verified using a digital simulation of a continuous system/microprocessor controller.**

I. Introduction

The primary impetus for applying LQG control concepts is improved system performance combined with the advent of digital electronic control implementation. Digital electronics provide the means by which complex controllers associated with LQG theory can be implemented. The current trend toward increased use of digital electronics—in particular, microprocessors—will lead to increased use of modern control logic including system identification, modeling, estimation, and multivariable control methodologies [1].

However, prior to widespread use of microprocessors for modern control logic implementation, key issues associated with microprocessor implementation must be addressed and resolved. These issues include 1) accuracy, 2) computational capability including arithmetic as well as interface speed, and 3) memory requirements [2]. Defining these requirements will establish criteria for selecting the appropriate computer system for control implementation. Consequently, this study was directed toward establishing microprocessor requirements for implementing continuous-time LQG control logic so that system performance with a microprocessor controller is close to system performance with the optimal analog controller.

II. Control of Linear Stochastic Systems

Open-loop linear time-invariant continuous system dynamics are described by the differential and algebraic equations

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + \xi(t) \\ y(t) &= Cx(t) + Du(t) \\ z(t) &= Ex(t) + \eta(t)\end{aligned} \quad (1)$$

where x, u, y, and z represent n states, m inputs, p outputs, and l measurements, respectively. The random process vectors ξ and η represent white zero-mean Gaussian noise. Under appropriate conditions the optimal input u^* exists [3] and is described by

$$\begin{aligned}\dot{\hat{x}}(t) &= F\hat{x}(t) + H(z(t) - E\hat{x}(t)) \\ u^*(t) &= G\hat{x}(t) \\ F &\triangleq A + BG\end{aligned} \quad (2)$$

where the notation (^) denotes the estimate of the variable in parentheses and G and H represent the deterministic feedback control gains and steady-state Kalman filter gains, respectively.

Equation (2) indicates that the optimal input is a continuous function. To implement this control law on digital electronics the control law must be discretized. Note that the regulator control problem may be for-

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