

# RESPONSE OF A STRATIFIED WATER BODY TO WIND

## Surface set-up and interfacial displacement

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Sem. Otoño 2004

Consider a water body of finite extension  $L$  in the  $x$  direction, that is stratified in two layers of thicknesses  $h_1$  and  $h_2$ , respectively, with densities  $\rho_1$  and  $\rho_2$ , respectively, where the subindex 1 denotes the upper layer and the subindex 2 the lower layer (Fig. 1). For stable stratification,  $\rho_1 < \rho_2$ , obviously. Consider now wind blowing over the free surface in the  $x$  direction. A shear stress  $\tau_s$  is thus exerted on the free surface in the same direction. This shear stress is equivalent to a vertical transfer of longitudinal momentum to the initially still water volume and a flow is established within that volume in response.

Applying the Reynolds averaged Navier-Stokes equations to each layer of Fig. 1, assuming flow only in the  $x - z$  plane, yields:

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} + w_i \frac{\partial u_i}{\partial z} = -\frac{1}{\rho_i} \left( \frac{\partial \hat{p}}{\partial x} \right)_i + \frac{1}{\rho_i} \frac{\partial (\tau_{xx})_i}{\partial x} + \frac{1}{\rho_i} \frac{\partial (\tau_{zx})_i}{\partial z} \quad (1)$$

$$\frac{\partial w_i}{\partial t} + u_i \frac{\partial w_i}{\partial x} + w_i \frac{\partial w_i}{\partial z} = -\frac{1}{\rho_i} \left( \frac{\partial \hat{p}}{\partial z} \right)_i + \frac{1}{\rho_i} \frac{\partial (\tau_{xz})_i}{\partial x} + \frac{1}{\rho_i} \frac{\partial (\tau_{zz})_i}{\partial z} \quad (2)$$

where  $u_i$  and  $w_i$  denote the horizontal and vertical components of flow velocity induced by the wind in layer  $i$ , respectively, with  $i = 1, 2$ ;  $\hat{p}_i$  denotes the piezometric pressure in layer  $i$ ;  $(\tau_{xx})_i$  and  $(\tau_{zz})_i$  denote normal stresses in layer  $i$ , in the longitudinal and vertical directions, respectively; and  $(\tau_{xz})_i$  and  $(\tau_{zx})_i$  denote shear stresses in layer  $i$ , in the longitudinal and vertical directions, respectively.

Applying the continuity equation for incompressible fluid to each layer yields:

$$\frac{\partial u_i}{\partial x} + \frac{\partial w_i}{\partial z} = 0 \quad (3)$$

Neglecting non-linear terms in the left hand side of equations (1) and (2), assuming as a first order approximation that the wind induced flow velocities are small, and that the pressure in each layer is hydrostatic, those equations are reduced to:

$$\frac{\partial u_i}{\partial t} = -\frac{1}{\rho_i} \left( \frac{\partial \hat{p}}{\partial x} \right)_i + \frac{1}{\rho_i} \frac{\partial (\tau_{zx})_i}{\partial z} \quad (4)$$

$$\left( \frac{\partial \hat{p}}{\partial z} \right)_i = 0 \quad (5)$$

Suppose the wind induces displacements  $\xi_1$  and  $\xi_2$  of the free surface and density interface between layers 1 and 2, respectively (Fig. 2). In each layer the pressure is hydrostatic, therefore:

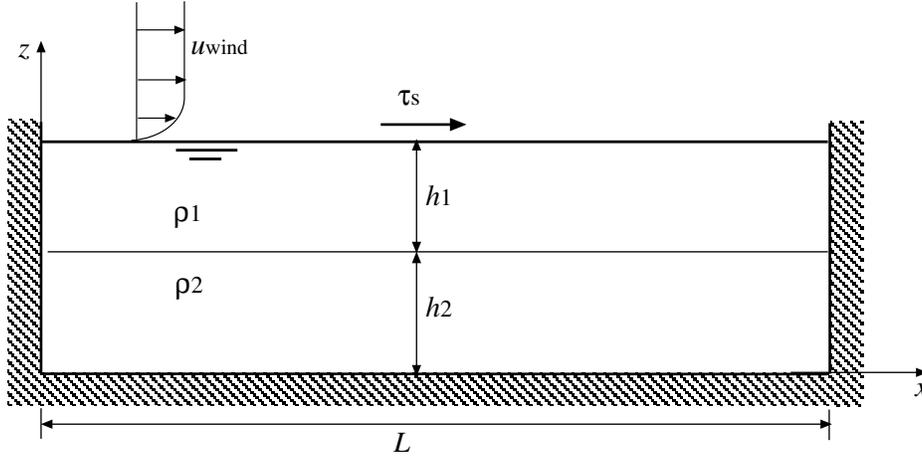


Figure 1: Two-layer stratified fluid with wind shear stress acting in the  $x$  direction.

$$\hat{p}_i = p_i + \rho_i g z = \text{constant} \quad (6)$$

where  $p_i$  denotes the thermodynamic pressure in layer  $i$ , which varies with  $x$  and  $z$ . It is easy to see that for points  $A$  and  $B$  in Fig. 3:  $p_A = 0$  and  $p_B = \rho_1 g (h_1 + \xi_1 - \xi_2)$ . With this result, the respective piezometric pressures are then:

$$\hat{p}_1 = \hat{p}_A = \rho_1 g (h_1 + h_2 + \xi_1) \quad (7)$$

$$\hat{p}_2 = \hat{p}_B = \rho_1 g (h_1 + \xi_1 - \xi_2) + \rho_2 g (h_2 + \xi_2) \quad (8)$$

from where the longitudinal piezometric pressure gradients in layers 1 and 2 are obtained:

$$\left(\frac{\partial \hat{p}}{\partial x}\right)_1 = \rho_1 g \frac{\partial \xi_1}{\partial x} \quad (9)$$

$$\left(\frac{\partial \hat{p}}{\partial x}\right)_2 = \rho_1 g \frac{\partial \xi_1}{\partial x} + (\rho_2 - \rho_1) g \frac{\partial \xi_2}{\partial x} \quad (10)$$

Replacing these expressions in equation (4) yields, for each layer:

$$\frac{\partial u_1}{\partial t} = -g \frac{\partial \xi_1}{\partial x} + \frac{1}{\rho_1} \frac{\partial (\tau_{zx})_1}{\partial z} \quad (11)$$

$$\frac{\partial u_2}{\partial t} = -g \frac{\rho_1}{\rho_2} \frac{\partial \xi_1}{\partial x} - g \frac{(\rho_2 - \rho_1)}{\rho_2} \frac{\partial \xi_2}{\partial x} + \frac{1}{\rho_2} \frac{\partial (\tau_{zx})_2}{\partial z} \quad (12)$$

These equations are now depth-averaged by integrating vertically within each layer, to obtain equations that predict the time evolution of the depth-averaged wind induced velocity in each layer:

$$\int_{h_2}^{h_1+h_2} \frac{\partial u_1}{\partial t} dz = -g \int_{h_2}^{h_1+h_2} \frac{\partial \xi_1}{\partial x} dz + \frac{1}{\rho_1} \int_{h_2}^{h_1+h_2} \frac{\partial (\tau_{zx})_1}{\partial z} dz \quad (13)$$

$$\int_0^{h_2} \frac{\partial u_2}{\partial t} dz = -g \frac{\rho_1}{\rho_2} \int_0^{h_2} \frac{\partial \xi_1}{\partial x} dz - g \frac{(\rho_2 - \rho_1)}{\rho_2} \int_0^{h_2} \frac{\partial \xi_2}{\partial x} dz + \frac{1}{\rho_2} \int_0^{h_2} \frac{\partial (\tau_{zx})_2}{\partial z} dz \quad (14)$$

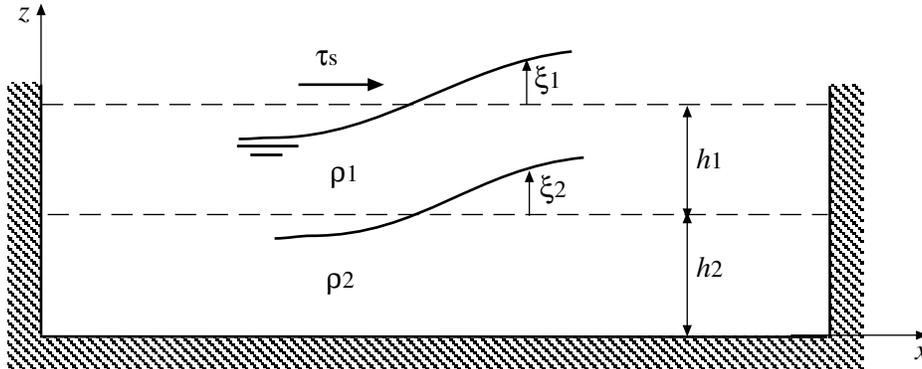


Figure 2: Definition of free surface and density interface displacements,  $\xi_1$  y  $\xi_2$ , respectively.

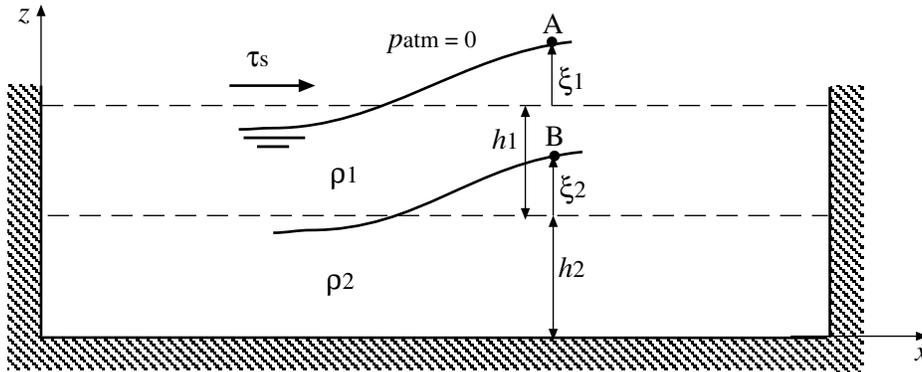


Figure 3: Determining the piezometric pressure in layers 1 and 2.

Considering that  $\xi_1$  and  $\xi_2$  are independent of  $z$  and defining the depth-averaged velocities in each layer as:

$$U_1 = \frac{1}{h_1} \int_{h_2}^{h_1+h_2} u_1 dz \tag{15}$$

$$U_2 = \frac{1}{h_2} \int_0^{h_2} u_2 dz \tag{16}$$

yields:

$$\frac{\partial U_1}{\partial t} = -g \frac{\partial \xi_1}{\partial x} + \frac{1}{\rho_1 h_1} ((\tau_{zx})_{h_1+h_2} - (\tau_{zx})_{h_2}) \tag{17}$$

$$\frac{\partial U_2}{\partial t} = -g \frac{\rho_1}{\rho_2} \frac{\partial \xi_1}{\partial x} - g \frac{(\rho_2 - \rho_1)}{\rho_2} \frac{\partial \xi_2}{\partial x} + \frac{1}{\rho_2 h_2} ((\tau_{zx})_{h_2} - (\tau_{zx})_0) \tag{18}$$

In the depth-averaged equations, the only relevant shear stresses are those acting on the free surface,  $(\tau_{zx})_{h_1+h_2} = \tau_s$ , on the density interface,  $(\tau_{zx})_{h_2} = \tau_i$ , and on the bottom wall,  $(\tau_{zx})_0 = \tau_f$ .

The system of equations (17) and (18) is similar to the classic result proposed by Spigel and Imberger (1980) and Heaps (1984).

The steady state situation is attained when  $\partial U_i/\partial t = 0$ , that is, when the wind induced flow in each layer is completely developed. In such situation, equations (17) and (18) yield expressions to evaluate the steady state slopes of the free surface and density interface displacements:

$$\frac{\partial \xi_1}{\partial x} = \frac{1}{g\rho_1 h_1} (\tau_s - \tau_i) \quad (19)$$

$$\frac{\partial \xi_2}{\partial x} = -\frac{1}{g(\rho_2 - \rho_1)} \left( \frac{\tau_s}{h_1} - \tau_i \left( \frac{h_1 + h_2}{h_1 h_2} \right) + \frac{\tau_b}{h_2} \right) \quad (20)$$

Since the wind induced flow is driven by the surface shear,  $\tau_s$ , it is convenient to express the previous result as:

$$\frac{\partial \xi_1}{\partial x} = \frac{\tau_s}{g\rho_1 h_1} \left( 1 - \frac{\tau_i}{\tau_s} \right) \quad (21)$$

$$\frac{\partial \xi_2}{\partial x} = -\frac{\rho_1}{(\rho_2 - \rho_1)} \frac{\tau_s}{g\rho_1 h_1} \left( 1 - \frac{\tau_i}{\tau_s} \left( \frac{h_1 + h_2}{h_2} \right) + \frac{\tau_b}{\tau_s} \frac{h_1}{h_2} \right) \quad (22)$$

Defining the wind shear velocity as:  $u_{*s} = \sqrt{\tau_s/\rho_1}$ , then these equations can be rewritten as:

$$\frac{\partial \xi_1}{\partial x} = \frac{u_{*s}^2}{gh_1} \left( 1 - \frac{\tau_i}{\tau_s} \right) \quad (23)$$

$$\frac{\partial \xi_2}{\partial x} = -\frac{\rho_1}{(\rho_2 - \rho_1)} \frac{u_{*s}^2}{gh_1} \left( 1 - \frac{\tau_i}{\tau_s} \left( \frac{h_1 + h_2}{h_2} \right) + \frac{\tau_b}{\tau_s} \frac{h_1}{h_2} \right) \quad (24)$$

From this result two different cases can be analyzed:

### Case 1: Water basin with no stratification

This case corresponds to a one-layer water basin of depth  $h_1$  and constant density  $\rho_1$ . Wind induced shear tilts the free surface with a positive slope  $d\xi_1/dx$ , which increases the water surface elevation at the downwind end of the basin and lowers it at the upwind end due to conservation of volume (Fig. 4). This response is known as *wind set-up*.

Associated with the set-up of the free surface there is an oscillatory motion of the whole system, consisting of a periodic variation of the water surface slope that gives rise to a periodic alternation of the water surface elevation at the end walls of the basin. This motion is known as *seiche* and is generated particularly when the wind stops. A specific analysis of this motion will be addressed in a different lecture note.

In a one-layer basin, equation (23) changes to:

$$\frac{\partial \xi_1}{\partial x} = \frac{u_{*s}^2}{gh_1} \left( 1 - \frac{\tau_b}{\tau_s} \right) \quad (25)$$

However, it can be shown, both theoretically and experimentally, that the bottom shear stress,  $\tau_b$ , is generally small compared with  $\tau_s$ , the ratio of the two being of the order of 1 to 4 %, and therefore it can be neglected in the previous equation. With this assumption, the slope of the free surface is given by:

$$\frac{\partial \xi_1}{\partial x} = \frac{u_{*s}^2}{gh_1} \quad (26)$$

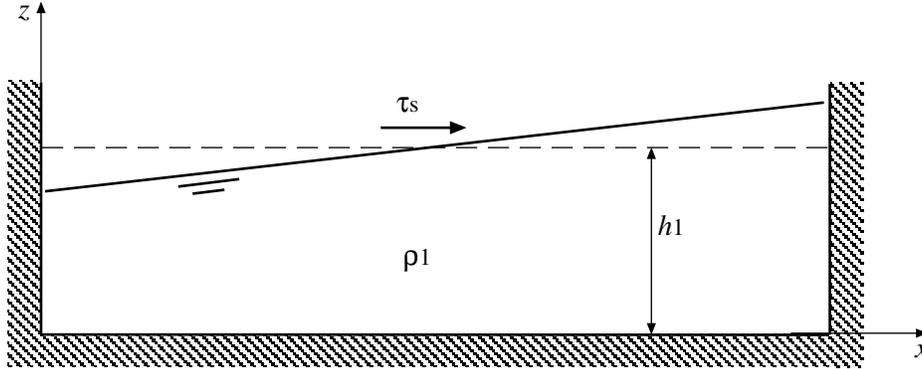


Figure 4: Wind set-up in a non-stratified water basin.

The right hand side of this equation has the structure of a Froude number. Defining  $Fr_* = u_{*s}/\sqrt{gh_1}$ , then the free surface slope is simply expressed as:

$$\frac{\partial \xi_1}{\partial x} = Fr_*^2 \quad (27)$$

### Case 2: Two-layer stratified basin

This is the case for which equations (23) and (24) were deduced. Usually it is assumed that the interfacial shear stress,  $\tau_i$ , just as the bottom shear stress, is negligible compared with  $\tau_s$ . With this classical assumption (Spigel and Imberger, 1980) the free surface and interfacial slopes associated with the response of the stratified water basin to wind, are given by:

$$\frac{\partial \xi_1}{\partial x} = \frac{u_{*s}^2}{gh_1} \quad (28)$$

$$\frac{\partial \xi_2}{\partial x} = -\frac{\rho_1}{(\rho_2 - \rho_1)} \frac{u_{*s}^2}{gh_1} \quad (29)$$

or, introducing the following definition:

$$Ri_0 = \frac{(\rho_2 - \rho_1)}{\rho_1} \frac{gh_1}{u_{*s}^2} = \frac{\Delta\rho}{\rho_1} \frac{1}{Fr_*^2} \quad (30)$$

where  $Ri_0$  denotes the dimensionless parameter known as *Richardson Number*:

$$\frac{\partial \xi_1}{\partial x} = Fr_*^2 \quad (31)$$

$$\frac{\partial \xi_2}{\partial x} = -\frac{1}{Ri_0} \quad (32)$$

From these expressions it is concluded that the free surface tilt of the two-layer stratified basin is identical to that of the one-layer non-stratified basin. It can also be concluded that the density interface has a negative slope, indicating that the tilt is opposite to that of the free surface, with a

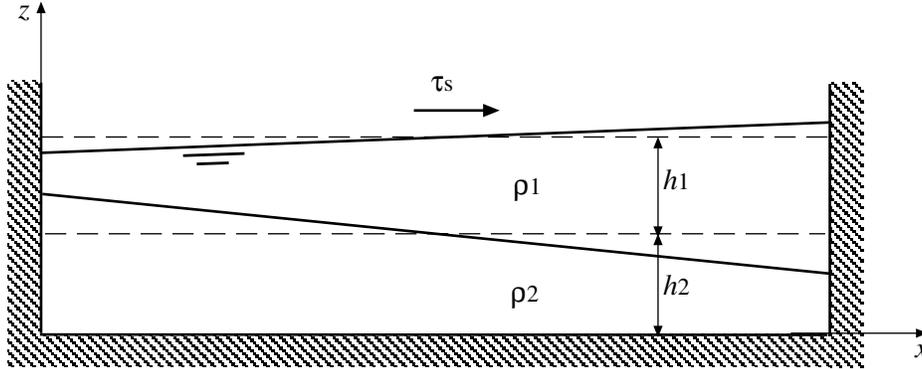


Figure 5: Wind induced free surface and density interface tilt in a two-layer stratified basin.

lower interface elevation at the downwind end of the basin and a higher interface elevation at the upwind end (Fig. 5). The ratio of the interface to free surface slopes results to be:

$$\frac{\partial \xi_2 / \partial x}{\partial \xi_1 / \partial x} = -\frac{\rho_1}{\Delta \rho} \quad (33)$$

which, given that the density difference between the surface and bottom layers,  $\Delta \rho$ , is generally small, indicates that the inclination of the density interface is much larger than that of the free surface and that the latter can ultimately be neglected with respect to the former.

Just as it occurs with the free surface, the density interface can also undergo an oscillatory motion known as *internal seiche*. This behavior will be analyzed in a different lecture note.

It is interesting to note that the Richardson number  $Ri_0$  is inversely proportional to the wind induced surface shear stress. This means, from (32), that large wind speeds are associated with large interfacial slopes or important tilts of the density interface. Given that to satisfy volume conservation the interface pivots around the middle point along the basin, a situation may occur, at large enough wind speeds, for which the density interface goes all the way up to the free surface at the upwind end of the water basin (Fig. 6). This phenomenon is known as *upwelling* and its occurrence implies a much more complex behavior of the hydrodynamics of the system, since the density gradients have in this case important horizontal components. The existence of upwelling may force the use of 2-D or 3-D models to predict the behavior of the stratification in the system, instead of 1-D models that are only able to capture vertical variations of the stratification. Besides, the occurrence of upwelling has important environmental consequences, as it implies the irruption in the surface layer of colder water from the hypolimnion with typically worse quality than the original surface water.

For a basin of longitudinal extension  $L$ , the interfacial slope predicted by (32) gives a condition for upwelling to occur:

$$\frac{h_1}{L} Ri_0 \approx \frac{1}{2} \quad (34)$$

The dimensionless parameter  $W = (h_1/L) Ri_0$ , is known as *Wedderburn Number* (Imberger and Patterson, 1989). It provides a criterion to determine whether a given wind event will generate

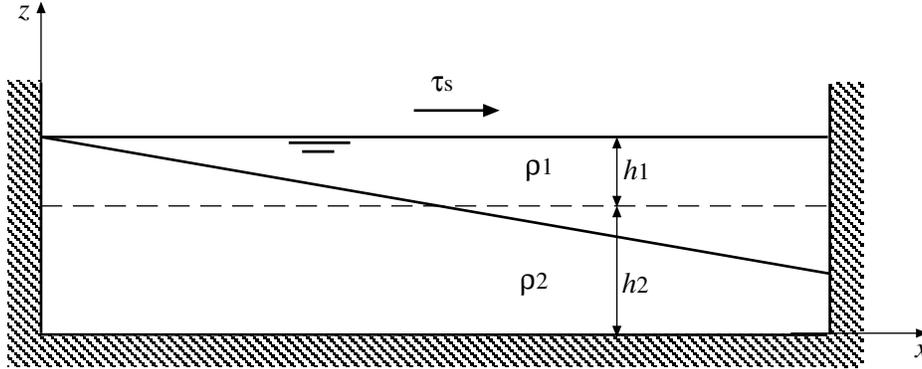


Figure 6: Upwelling due to strong tilting of the density interface associated to strong winds and shallow surface layer. Free surface set-up has been neglected for simplicity.

upwelling. In practice, a threshold value  $W = 1$  is used, such that upwelling will tend to occur for values of the Wedderburn number lower than unity.

The concept of the Wedderburn number as a criterion for upwelling has been generalized by Imberger and Patterson (1989). They define a different parameter called *Lake Number*, which is computed taking into account a continuous stratification,  $\rho(z)$ , of the water body, not only the two-layer case as in the analysis made here so far. As the wind stress is exerted on the free surface, a net force acts that tend to overturn the density structure of the water column. At critical equilibrium, the moment exerted by the wind about the center of volume of the water body (located at an elevation  $z_v$ ) will be exactly balanced by a restoring moment exerted by gravity acting on the center of mass (located at an elevation  $z_m$ ). Due to the stratification:  $z_m < z_v$ . The critical equilibrium condition is given by:

$$\int_{A_s} \rho_s u_{*s}^2 dA (z_s - z_v) = M g (z_v - z_m) \sin \beta \quad (35)$$

where  $A_s$  is the surface area where a generally variable wind shear velocity,  $u_{*s}$ , is applied;  $\rho_s$  denotes the density of surface water;  $z_s$  is the elevation of the free surface;  $M$  is the total mass of the stratified water body and  $\beta$  is the angle with respect to the vertical subtended by the centers of volume and mass (Fig. 7).

If the wind is not strong enough to cause upwelling, the critical equilibrium condition is not met and the restoring moment of gravity is larger than the overturning moment exerted by the wind. The Lake number is defined as:

$$L_N = \frac{M g (z_v - z_m) \sin \beta}{\int_{A_s} \rho_s u_{*s}^2 dA (z_s - z_v)} \quad (36)$$

The geometry and density structure of the water body define  $M$ , the total volume,  $V$ ,  $z_m$  and  $z_v$ :

$$M = \int_0^{z_s} \rho(z) A(z) dz \quad (37)$$

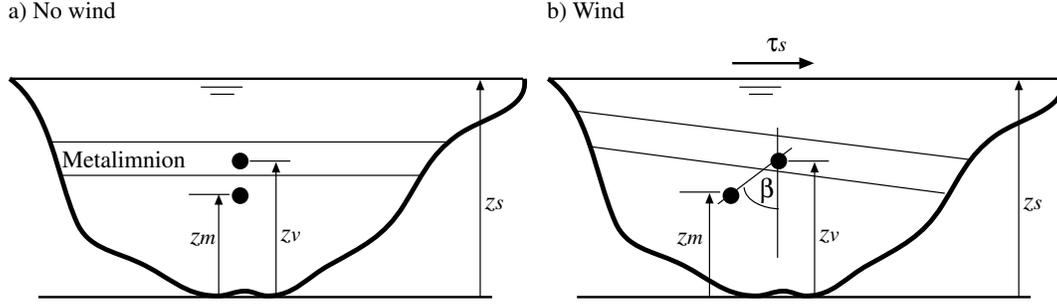


Figure 7: Change in location of center of mass of a stratified water body due to wind induced surface shear stress.

$$V = \int_0^{z_s} A(z) dz \quad (38)$$

$$z_m = \frac{\int_0^{z_s} z \rho(z) A(z) dz}{M} \quad (39)$$

$$z_v = \frac{\int_0^{z_s} z A(z) dz}{V} \quad (40)$$

Replacing these definitions in (36), assuming that  $u_{*s}$  is uniform across the surface of the water body, gives:

$$L_N = \frac{g S_t \sin \beta}{\rho_s u_{*s}^2 A_s (z_s - z_v)} \quad (41)$$

where  $S_t$  given by:

$$S_t = M (z_v - z_m) = \int_0^{z_s} (z_v - z) \rho(z) A(z) dz \quad (42)$$

is called the *stability* of the water body.

The angle  $\beta$  can be estimated for the critical condition as that for which the metalimnion intersects the free surface, so that:

$$\beta = \frac{z_s - z_T}{L/2} \quad (43)$$

where  $z_T$  denotes the elevation of the center of the metalimnion (where the hypothetical thermocline would be located), and  $L$  is the fetch of the wind in the lake. The fetch is scaled using the surface area, such that:  $L/2 \approx A_s^{1/2}$ . With this result, the Lake number is given by:

$$L_N = \frac{g S_t (1 - z_T/z_s)}{\rho_s u_{*s}^2 A_s^{3/2} (1 - z_v/z_s)} \quad (44)$$

For large values of the Lake number the stratification is severe and dominate the forces induced by the wind stress. Under these circumstances, stratification is expected to be mainly horizontal. On the contrary, small values of the Lake number are associated with strong winds and weak stratification, a situation for which upwelling may occur.

**References:**

Heaps, N. S. (1984). "Hydrodynamics of Lakes". K. Hutter, editor. Springer-Verlag.

Imberger, J. y Patterson, J. C. (1989). "Physical Limnology". Adv. Applied Mech., Vol. 27, pp. 303-475.

Spigel, R. H. e Imberger, J. (1980). "The classification of mixed-layer dynamics in lakes of small to medium size". J. Phys. Oceanogr., 10, pp. 1104-1121.