SURFACE AND INTERNAL WAVES
Linear Theory for Long Waves of Small Amplitude

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Sem. Otoño 2004

Consider a two-layer stratified water basin. The upper and bottom layers have thicknesses h_1 and h_2, respectively, and densities ρ_1 and ρ_2, respectively (Fig. 1). For stable stratification, $\rho_1 < \rho_2$, obviously. Consider also the free surface and density interface deformation due to the presence of long waves of small amplitude. These waves correspond to a free response of the system, in absence of any forcing such as that exerted by the surface shear stress induced by the wind. The waves can be generated, for instance, at the end of a wind event that created the set-up of the free surface and the downward tilt of the density interface, due to the relaxation of the surface shear stress. In such situation both the free surface and the density interface undergo a free oscillatory motion, associated with the relaxation of their respective forced deformation.

Applying the Reynolds averaged Navier-Stokes equations to each layer of Fig. 1, assuming 2-D flow in the $x - z$ plane, yields:

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} + w_i \frac{\partial u_i}{\partial z} = -\frac{1}{\rho_i}(\frac{\partial \hat{p}}{\partial x})_i + \frac{1}{\rho_i}\frac{\partial (\tau_{xx})_i}{\partial x} + \frac{1}{\rho_i}\frac{\partial (\tau_{zx})_i}{\partial z} \tag{1}$$

$$\frac{\partial w_i}{\partial t} + u_i \frac{\partial w_i}{\partial x} + w_i \frac{\partial w_i}{\partial z} = -\frac{1}{\rho_i}(\frac{\partial \hat{p}}{\partial z})_i + \frac{1}{\rho_i}\frac{\partial (\tau_{xz})_i}{\partial x} + \frac{1}{\rho_i}\frac{\partial (\tau_{zz})_i}{\partial z} \tag{2}$$

where u_i and w_i denote the horizontal and vertical components of the flow velocity in layer i, with $i = 1, 2$, $\hat{p}_i$ denotes piezometric pressure in layer i, $(\tau_{xx})_i$ and $(\tau_{zz})_i$ denote normal stresses in layer i
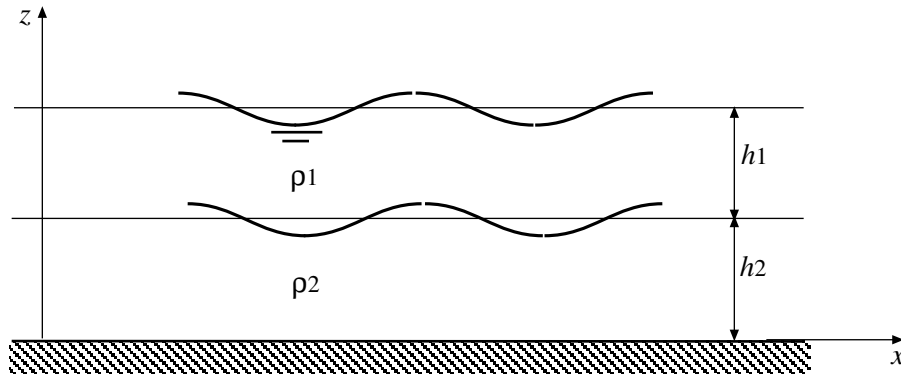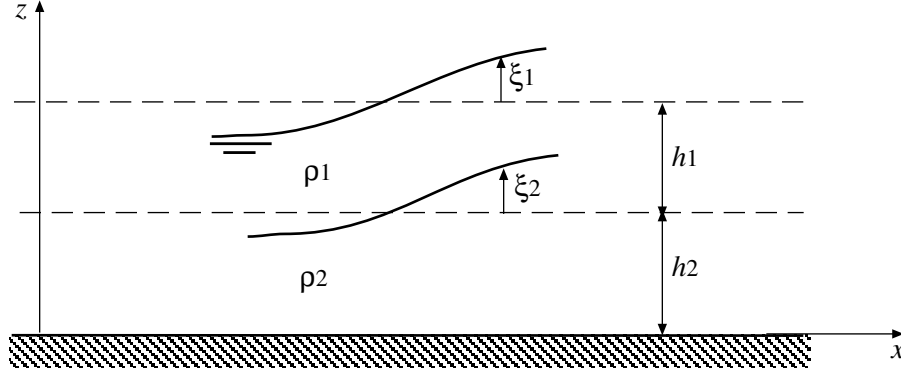

Figure 1: Two-layer stratified water basin.

Figure 2: Definition of free surface and density interface displacements, ξ_1 and ξ_2, respectively.

in directions x and z, respectively, and $(\tau_{xz})_i$ and $(\tau_{zx})_i$ denote shear stresses in layer i in directions x and z, respectively.

The continuity equation in each layer reads:

$$\frac{\partial u_i}{\partial x} + \frac{\partial w_i}{\partial z} = 0 \tag{3}$$

Introducing boundary layer approximations, including assuming hydrostatic pressure in both layers, equations (1) and (2) reduce to:

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i^2}{\partial x} + \frac{\partial w_i u_i}{\partial z} = -\frac{1}{\rho_i}\left(\frac{\partial \hat{p}}{\partial x}\right)_i + \frac{1}{\rho_i}\frac{\partial (\tau_{zx})_i}{\partial z} \tag{4}$$

$$\left(\frac{\partial \hat{p}}{\partial z}\right)_i = 0 \tag{5}$$

Assume that the oscillations of the system induce vertical displacements ξ_1 and ξ_2 of the free surface and density interface, respectively, as shown in Fig. 2. Since pressure is assumed to be hydrostatic, then the following relationship applies in each layer i:

$$\hat{p}_i = p_i + \rho_i g z = constant. \tag{6}$$

where p_i denotes thermodynamic pressure in layer i, which varies with x and z. For points A and B in Fig. 3, $p_A = 0$ and $p_B = \rho_1 g(h_1 + \xi_1 - \xi_2)$. Using these results, the piezometric pressures are given by:

$$\hat{p}_1 = \hat{p}_A = \rho_1 g(h_1 + h_2 + \xi_1) \tag{7}$$

$$\hat{p}_2 = \hat{p}_B = \rho_1 g(h_1 + \xi_1 - \xi_2) + \rho_2 g(h_2 + \xi_2) \tag{8}$$

from where the longitudinal piezometric pressure gradients in layers 1 and 2 are obtained as:

$$\left(\frac{\partial \hat{p}}{\partial x}\right)_1 = \rho_1 g \frac{\partial \xi_1}{\partial x} \tag{9}$$

$$\left(\frac{\partial \hat{p}}{\partial x}\right)_2 = \rho_1 g \frac{\partial \xi_1}{\partial x} + (\rho_2 - \rho_1) g \frac{\partial \xi_2}{\partial x} \tag{10}$$
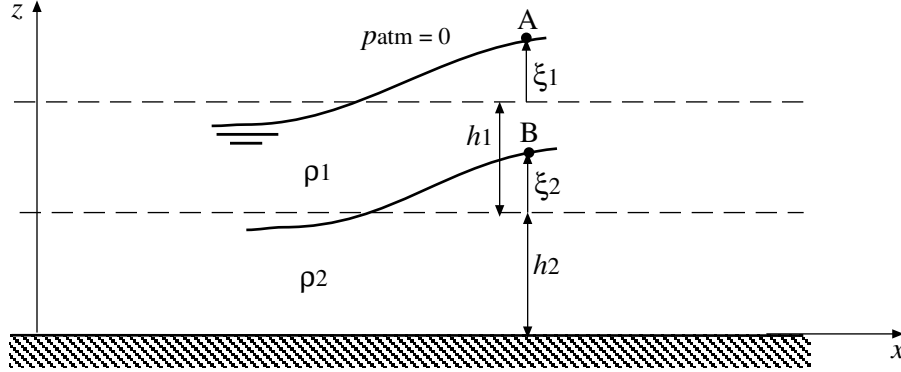
Figure 3: Determining the piezometric pressure gradient in layers 1 and 2.

Replacing these expressions in equation (4) yields, for each layer:

$$\frac{\partial u_1}{\partial t} + \frac{\partial u_1^2}{\partial x} + \frac{\partial w_1 u_1}{\partial z} = -g\frac{\partial \xi_1}{\partial x} + \frac{1}{\rho_1}\frac{\partial (\tau_{zx})_1}{\partial z} \tag{11}$$

$$\frac{\partial u_2}{\partial t} + \frac{\partial u_2^2}{\partial x} + \frac{\partial w_2 u_2}{\partial z} = -g\frac{\rho_1}{\rho_2}\frac{\partial \xi_1}{\partial x} - g\frac{(\rho_2 - \rho_1)}{\rho_2}\frac{\partial \xi_2}{\partial x} + \frac{1}{\rho_2}\frac{\partial (\tau_{zx})_2}{\partial z} \tag{12}$$

On the other hand, equation (3) for each layer is:

$$\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z} = 0 \tag{13}$$

$$\frac{\partial u_2}{\partial x} + \frac{\partial w_2}{\partial z} = 0 \tag{14}$$

Integrating these equations vertically in each layer, with the aim of obtaining equations governing the temporal evolution of the depth-averaged velocity in each layer, gives:

$$\int_{h_2+\xi_2}^{h_1+h_2+\xi_1} \frac{\partial u_1}{\partial t}dz + \int_{h_2+\xi_2}^{h_1+h_2+\xi_1} \frac{\partial u_1^2}{\partial x}dz + \int_{h_2+\xi_2}^{h_1+h_2+\xi_1} \frac{\partial w_1 u_1}{\partial z}dz =$$

$$-g\int_{h_2+\xi_2}^{h_1+h_2+\xi_1} \frac{\partial \xi_1}{\partial x}dz + \frac{1}{\rho_1}\int_{h_2+\xi_2}^{h_1+h_2+\xi_1} \frac{\partial (\tau_{zx})_1}{\partial z}dz \tag{15}$$

$$\int_0^{h_2+\xi_2} \frac{\partial u_2}{\partial t}dz + \int_0^{h_2+\xi_2} \frac{\partial u_2^2}{\partial x}dz + \int_0^{h_2+\xi_2} \frac{\partial w_2 u_2}{\partial z}dz =$$

$$-g\frac{\rho_1}{\rho_2}\int_0^{h_2+\xi_2} \frac{\partial \xi_1}{\partial x}dz - g\frac{(\rho_2 - \rho_1)}{\rho_2}\int_0^{h_2+\xi_2} \frac{\partial \xi_2}{\partial x}dz + \frac{1}{\rho_2}\int_0^{h_2+\xi_2} \frac{\partial (\tau_{zx})_2}{\partial z}dz \tag{16}$$

$$\int_{h_2+\xi_2}^{h_1+h_2+\xi_1} \frac{\partial u_1}{\partial x}dz + \int_{h_2+\xi_2}^{h_1+h_2+\xi_1} \frac{\partial w_1}{\partial z}dz = 0 \tag{17}$$

$$\int_0^{h_2+\xi_2} \frac{\partial u_2}{\partial x}dz + \int_0^{h_2+\xi_2} \frac{\partial w_2}{\partial z}dz = 0 \tag{18}$$

Applying the kinematic boundary condition at the free surface and density interface and the non-slip and non-penetration conditions at the bottom boundary, defining the depth averaged velocities in each layer as:

$$U_1 = \frac{1}{h_1} \int_{h_2+\xi_2}^{h_1+h_2+\xi_1} u_1 dz \tag{19}$$

$$U_2 = \frac{1}{h_2} \int_0^{h_2+\xi_2} u_2 dz \tag{20}$$

and neglecting non-linear terms in the left hand side of the resulting equations, assuming small depth-averaged velocities associated with the small amplitude surface and internal waves, yields:

$$\frac{\partial U_1}{\partial t} = -g\frac{\partial \xi_1}{\partial x} + \frac{1}{\rho_1 h_1}((\tau_{zx})_{h_1+h_2} - (\tau_{zx})_{h_2}) \tag{21}$$

$$\frac{\partial U_2}{\partial t} = -g\frac{\rho_1}{\rho_2}\frac{\partial \xi_1}{\partial x} - g\frac{(\rho_2 - \rho_1)}{\rho_2}\frac{\partial \xi_2}{\partial x} + \frac{1}{\rho_2 h_2}((\tau_{zx})_{h_2} - (\tau_{zx})_0) \tag{22}$$

$$h_1\frac{\partial U_1}{\partial x} + \frac{\partial \xi_1}{\partial t} - \frac{\partial \xi_2}{\partial t} = 0 \tag{23}$$

$$h_2\frac{\partial U_2}{\partial x} + \frac{\partial \xi_2}{\partial t} = 0 \tag{24}$$

The only relevant shear stresses for the depth averaged equations are those evaluated at the free surface, $(\tau_{zx})_{h_1+h_2} = \tau_s$, density interface, $(\tau_{zx})_{h_2} = \tau_i$, and bottom, $(\tau_{zx})_0 = \tau_b$. With these definitions the above system of equations can be rewritten as:

$$\frac{\partial U_1}{\partial t} + g\frac{\partial \xi_1}{\partial x} = \frac{1}{\rho_1 h_1}(\tau_s - \tau_i) \tag{25}$$

$$\frac{\partial U_1}{\partial x} + \frac{1}{h_1}\frac{\partial \xi_1}{\partial t} - \frac{1}{h_1}\frac{\partial \xi_2}{\partial t} = 0 \tag{26}$$

$$\frac{\partial U_2}{\partial t} + g\frac{\rho_1}{\rho_2}\frac{\partial \xi_1}{\partial x} + g\frac{(\rho_2 - \rho_1)}{\rho_2}\frac{\partial \xi_2}{\partial x} = \frac{1}{\rho_2 h_2}(\tau_i - \tau_b) \tag{27}$$

$$\frac{\partial U_2}{\partial x} + \frac{1}{h_2}\frac{\partial \xi_2}{\partial t} = 0 \tag{28}$$

The system of equations (25) to (28) governs the response of the two-layer stratified water basin, in terms of depth averaged velocities in each layer and deformation of the free surface and density interface, to shear stresses τ_s, τ_i and τ_b. Nonetheless, the same system of equations governs the free oscillations of the system when all the forcing shear stresses vanish. In such a case the right hand side of all equations (25)-(28) vanishes and the problem reduces to an eigenvalue problem.

From this result two possible situations are considered next.

Case 1: Non-stratified water basin

This case corresponds to a water basin of depth h_1 and constant density ρ_1. Free oscillations of this system correspond to the condition $\tau_s = \tau_b = 0$. The system of equations (25) to (28) reduces to:

$$\frac{\partial U_1}{\partial t} + g\frac{\partial \xi_1}{\partial x} = 0 \tag{29}$$

$$\frac{\partial U_1}{\partial x} + \frac{1}{h_1}\frac{\partial \xi_1}{\partial t} = 0 \tag{30}$$

To determine the normal modes of oscillation of the system assume a response in the form of a progressive wave, for which the depth-averaged velocity and free surface deformation are given by:

$$U_1 = \Upsilon_1 \sin(\alpha x - \omega t) \tag{31}$$

$$\xi_1 = \Xi_1 \sin(\alpha x - \omega t) \tag{32}$$

where $\alpha = 2\pi/\lambda$, $\omega = 2\pi/T$, and λ and T denote the wavelength and period of the surface waves, respectively. Likewise, Υ_1 and Ξ_1 denote the amplitudes of the velocity and surface deformation waves, respectively.

Introducing these definitions in (29) and (30), then the following algebraic problem for Υ_1 and Ξ_1 is obtained.

$$\begin{vmatrix} -\omega & \alpha g \\ \alpha & -\frac{\omega}{h_1} \end{vmatrix} \begin{vmatrix} \Upsilon_1 \\ \Xi_1 \end{vmatrix} = 0 \tag{33}$$

This is an eigenvalue problem, since in order to have a non-trivial (different from zero) solution for the amplitudes Υ_1 and Ξ_1, it is required for the matrix of coefficients in the previous equation to have a determinant equal to zero. This condition yields a *dispersion relationship* for the free oscillations of the system, that is, a relationship that relates the wave period T with a given wavelength λ. In fact, the condition: determinant of the coefficient matrix in (33) equal to zero, gives:

$$\frac{\omega^2}{h_1} - g\alpha^2 = 0 \tag{34}$$

from where the dispersion relationship results to be:

$$\frac{\omega}{\alpha} = \pm\sqrt{gh_1} \tag{35}$$

or,

$$c = \frac{\lambda}{T} = \pm\sqrt{gh_1} \tag{36}$$

where c denotes the celerity or displacement velocity of the surface waves.

This is a classic result, which in the case of open channel flow yields the definition of Froude number, allowing the distinction between subcritical and supercritical flow, depending on whether the flow velocity is lower or higher than that of the surface waves, respectively.
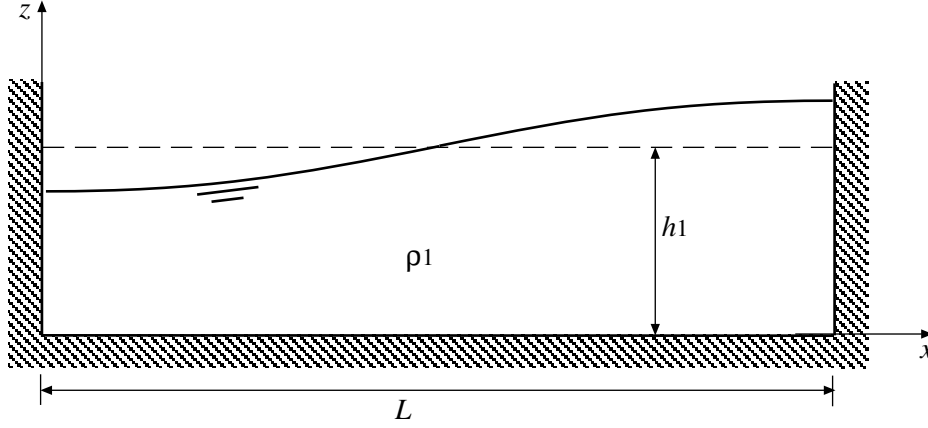
Figure 4: Surface seiche with wavelength $\lambda = 2L$.

From this result the period of oscillation of a surface seiche in a rectangular water basin of length L, can be determined. Such seiche can be considered to have a wavelength $\lambda = 2L$ (Fig. 4). In this case (36) yields:

$$T = \frac{2L}{\sqrt{gh_1}} \tag{37}$$

Case 2: Two-layer stratified water body

This is the case for which equations (25) to (28) were deduced. Imposing the condition $\tau_s = \tau_i = \tau_b = 0$ to analyze the free oscillations of the stratified system leads to:

$$\frac{\partial U_1}{\partial t} + g\frac{\partial \xi_1}{\partial x} = 0 \tag{38}$$

$$\frac{\partial U_1}{\partial x} + \frac{1}{h_1}\frac{\partial \xi_1}{\partial t} - \frac{1}{h_1}\frac{\partial \xi_2}{\partial t} = 0 \tag{39}$$

$$\frac{\partial U_2}{\partial t} + g\frac{\rho_1}{\rho_2}\frac{\partial \xi_1}{\partial x} + g\frac{(\rho_2 - \rho_1)}{\rho_2}\frac{\partial \xi_2}{\partial x} = 0 \tag{40}$$

$$\frac{\partial U_2}{\partial x} + \frac{1}{h_2}\frac{\partial \xi_2}{\partial t} = 0 \tag{41}$$

To determine the normal modes of oscillation, a response in terms of progressive waves can again be assumed, where the depth-averaged velocities in each layer and surface and interface deformations are given by:

$$U_i = \Upsilon_i \sin(\alpha x - \omega t) \tag{42}$$

$$\xi_i = \Xi_i \sin(\alpha x - \omega t) \tag{43}$$

where $i = 1, 2$ denote the surface and bottom layer, respectively. Just as in the one-layer case, $\alpha = 2\pi/\lambda$, $\omega = 2\pi/T$, and λ and T denote the wavelength and period of the oscillations of the
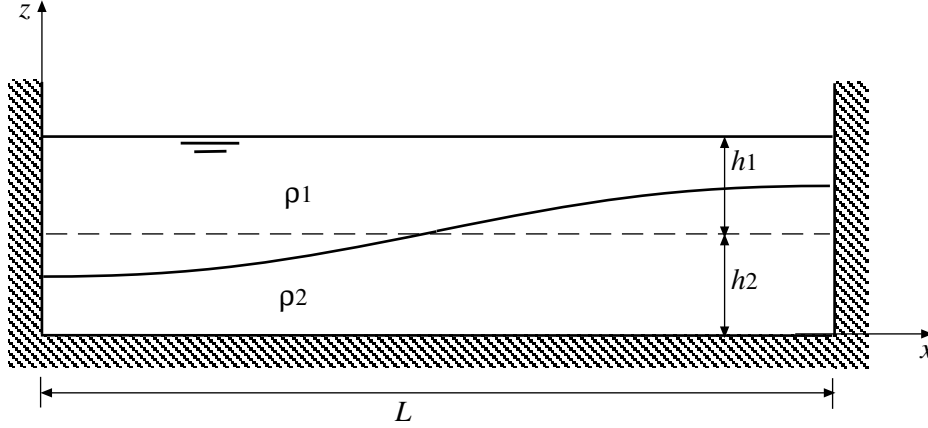
Figure 5: Internal seiche with wavelength $\lambda = 2L$.

system, respectively. Likewise, Υ_i and Ξ_i denote the amplitudes of the depth-averaged velocity and surface or interface deformation waves, in each layer i, respectively.

Replacing these definitions in the system of equations (38) to (41) yields the following algebraic problem for Υ_i and Ξ_i.

$$
\begin{vmatrix}
-\omega & \alpha g & 0 & 0 \\
\alpha & -\frac{\omega}{h_1} & 0 & \frac{\omega}{h_1} \\
0 & \alpha g \frac{\rho_1}{\rho_2} & -\omega & \alpha g \frac{(\rho_2 - \rho_1)}{\rho_2} \\
0 & 0 & \alpha & -\frac{\omega}{h_2}
\end{vmatrix}
\begin{vmatrix}
\Upsilon_1 \\
\Xi_1 \\
\Upsilon_2 \\
\Xi_2
\end{vmatrix} = 0
\tag{44}
$$

An eigenvalue problem is obtained again. In order to have a non-trivial solution for the amplitudes Υ_i and Ξ_i the matrix of coefficients in (44) must have a determinant equal to zero. Imposing such condition yields:

$$
-\frac{\alpha^2 \omega^2 g}{h_2} + \frac{\omega^4}{h_1 h_2} + \frac{\alpha^4 (\rho_2 - \rho_1) g^2}{\rho_2} - \frac{\alpha^2 \omega^2 (\rho_2 - \rho_1) g}{h_1 \rho_2} - \frac{\alpha^2 \omega^2 g \rho_1}{h_1 \rho_2} = 0
\tag{45}
$$

Simplifying the above equation, introducing the celerity of surface and internal waves: $c = \omega/\alpha$, leads to:

$$
\frac{c^4}{g(h_1 + h_2)} - c^2 + \frac{(\rho_2 - \rho_1)}{\rho_2} \left(\frac{h_1 h_2}{h_1 + h_2} \right) g = 0
\tag{46}
$$

which is an equation representing the dispersion relationship of long, small amplitude waves in a two-layer stratified water basin. Even though this equation has a simple analytical solution for c, it is convenient to note that the first term of the left hand side is of a lesser order of magnitude that the rest of the terms in the equation. Taking this into account, it can be shown that, without much error, c can be approximated by the relationship:

$$
c = \pm \sqrt{\frac{(\rho_2 - \rho_1)}{\rho_2} \left(\frac{h_1 h_2}{h_1 + h_2} \right) g}
\tag{47}
$$

This is also a classic results that shows that the celerity of interfacial waves in a stratified flow is much lower than that of surface waves in a non-stratified fluid, the ratio of both being of the order of $\sqrt{\Delta\rho/\rho_2}$, with $\Delta\rho = \rho_2 - \rho_1$, which is indeed a small number given the small value of the density difference $\Delta\rho$ usually observed in stratified water bodies such as lakes and reservoirs.

From this result the period of oscillation of an internal seiche in a rectangular basin of length L can be determined. Such seiche has a wavelength $\lambda = 2L$ (Fig. 5). In this case, from (47) T is given by:

$$T = \frac{2L}{\sqrt{\frac{\Delta\rho}{\rho_2} \frac{h_1 h_2}{h_1 + h_2} g}} \tag{48}$$

References:

Heaps, N. S. (1984). "Hydrodynamics of Lakes". K. Hutter, editor. Springer-Verlag.

Hutter, K. y Vischer, D. (1983). "Lake Hydraulics". P. Novak, editor. Chapter 1 in Developments in Hydraulic Engineering - 4. Elsevier Applied Science.

Imberger, J. y Patterson, J. C. (1990). "Physical Limnology". Adv. Applied Mech., Vol. 27, pp. 303-475.