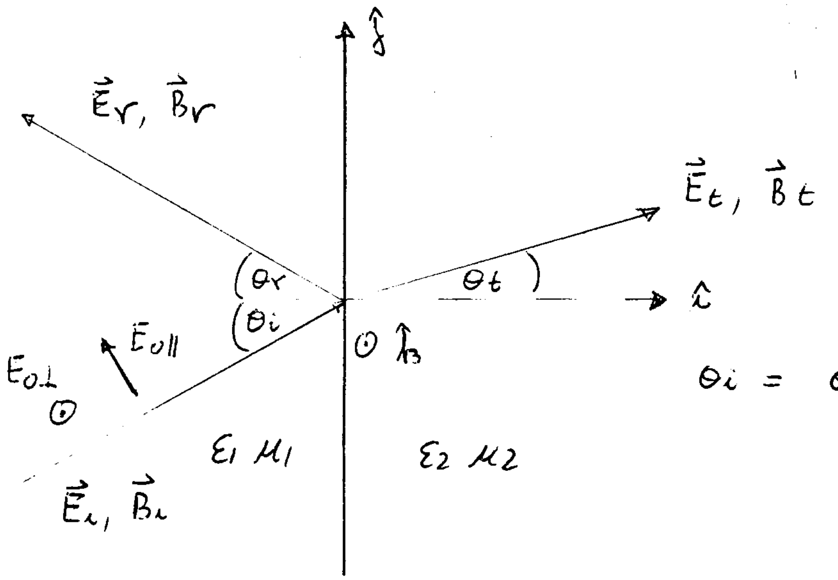
Reflexión y Refracción de Ondas

(1)



$\theta_i = \theta$ (dato)

$$\vec{E}_i = [E_{0\perp} \hat{k} + E_{0\parallel} (\cos\theta \hat{j} - \operatorname{sen}\theta \hat{\imath})] e^{j(\omega_i t - \vec{K}_i \cdot \vec{r})}$$

$$\vec{K}_i = K (\cos\theta \hat{\imath} + \operatorname{sen}\theta \hat{j}) \; ; \quad |K_i|/\omega_i = \sqrt{\varepsilon_1 \mu_1}$$

$$\vec{B}_i = \frac{\vec{K}_i}{\omega_i} \times \vec{E}_i = \sqrt{\varepsilon_1 \mu_1} \left[E_{0\parallel} \hat{k} + E_{0\perp} (\operatorname{sen}\theta \hat{\imath} - \cos\theta \hat{j}) \right] e^{j(\omega_i t - \vec{K}_i \cdot \vec{r})}$$

$$\vec{E}_r = \vec{E}_{0r} e^{j(\omega_r t - \vec{K}_r \cdot \vec{r})} \qquad \vec{K}_r = K_{rx} \hat{\imath} + K_{ry} \hat{j} + K_{rz} \hat{k}$$

$$\vec{B}_r = \frac{\vec{K}_r}{\omega_r} \times \vec{E}_r \qquad |\vec{K}_r|/\omega_r = \sqrt{\varepsilon_1 \mu_1}$$

$$\vec{E}_t = \vec{E}_{0t} e^{j(\omega_t t - \vec{K}_t \cdot \vec{r})} \qquad \vec{K}_t = K_{tx} \hat{\imath} + K_{ty} \hat{j} + K_{tz} \hat{k}$$

$$\vec{B}_t = \frac{\vec{K}_t}{\omega_t} \times \vec{E}_t \qquad |\vec{K}_t|/\omega_t = \sqrt{\varepsilon_2 \mu_2}$$

Condiciones de borde en $x = 0$

(1) $\hat{\imath} \times \vec{E}_1 = \hat{\imath} \times \vec{E}_2$ $\qquad \vec{E}_1 = \vec{E}_i + \vec{E}_r$

(2) $\hat{\imath} \times \vec{H}_1 = \hat{\imath} \times \vec{H}_2$ $\qquad \vec{E}_2 = \vec{E}_t$

(3) $D_{n1} = D_{n2}$ $\qquad \vec{H}_1 = \dfrac{\vec{B}_1}{\mu_1}$

(4) $B_{n1} = B_{n2}$ $\qquad \vec{H}_2 = \dfrac{\vec{B}_2}{\mu_2}$

Tomando la ecuación ① se tiene: (2)

$$\hat{\imath} \times \vec{E}_{0i}\, e^{j(\omega_i t - y K \operatorname{sen}\theta)} + \hat{\imath} \times \vec{E}_{0r}\, e^{j(\omega_r t - y K_{ry} - z K_{rz})}$$

$$= \hat{\imath} \times \vec{E}_{0t}\, e^{j(\omega_r t - y K_{ty} - z K_{tz})} \quad \text{(para todo } t, y, z)$$

Esto solo es posible si:

i) $\omega_i = \omega_r = \omega_t = \omega$

ii) $K_{iy} = K_{ry} = K_{ty} = K \operatorname{sen}\theta$

iii) $K_{iz} = K_{rz} = K_{tz} = 0$

de aquí se tiene lo siguiente:

a) $$\frac{|\vec{K}_r|}{\omega} = \frac{|\vec{K}_i|}{\omega} = \sqrt{\varepsilon_1 \mu_1} \Rightarrow K_{ry}^2 + K_{rx}^2 = K^2$$

$$K_{rx}^2 = K^2 - K_{ry}^2 = K^2(1 - \operatorname{sen}^2\theta) = K^2 \cos^2\theta$$

luego $K_{rx} = -K\cos\theta$ (onda reflejada en dirección $-\hat{\imath}$)

$$\operatorname{sen}\theta_r = \frac{K_{ry}}{|\vec{K}_r|} = \frac{K \operatorname{sen}\theta}{K} = \operatorname{sen}\theta$$

$\Rightarrow$ $\boxed{\theta_r = \theta}$ (Ley de reflexión de ángulos)

b) $$\frac{|\vec{K}_t|}{\omega} = \sqrt{\varepsilon_2 \mu_2}\,; \quad \frac{\operatorname{sen}\theta_t}{\operatorname{sen}\theta} = \frac{K_{ty}}{|\vec{K}_t|} \cdot \frac{|\vec{K}_i|}{K_{iy}}$$

$$\frac{\operatorname{sen}\theta_t}{\operatorname{sen}\theta} = \frac{K \operatorname{sen}\theta}{\omega\sqrt{\varepsilon_2\mu_2}} \cdot \frac{K}{K\operatorname{sen}\theta} = \frac{K/\omega}{\sqrt{\varepsilon_2\mu_2}} = \frac{\sqrt{\varepsilon_1\mu_1}}{\sqrt{\varepsilon_2\mu_2}}$$

se define $n = \sqrt{\dfrac{\varepsilon\mu}{\varepsilon_0\mu_0}}$

luego $$\frac{\operatorname{sen}\theta_t}{\operatorname{sen}\theta} = \frac{n_1}{n_2}$$ (Ley de Snell)

Cálculo de amplitudes

Tanto $\vec{E}_{or}$ como $\vec{E}_{ot}$ son perpendiculares a los vectores $\vec{K}_r$ y $\vec{K}_t$ respectivamente.

luego
$$\vec{E}_{or} = E_{r\perp}\,\hat{k} + E_{r\parallel}\,(\cos\theta\,\hat{j} + \operatorname{sen}\theta\,\hat{\imath})$$
$$\vec{E}_{ot} = E_{t\perp}\,\hat{k} + E_{t\parallel}\,(\cos\theta_t\,\hat{j} - \operatorname{sen}\theta_t\,\hat{\imath})$$
$$\vec{H}_{or} = \sqrt{\frac{\varepsilon_1}{\mu_1}}\left(-E_{r\parallel}\,\hat{k} + E_{r\perp}\,(\cos\theta\,\hat{j} + \operatorname{sen}\theta\,\hat{\imath})\right)$$
$$\vec{H}_{ot} = \sqrt{\frac{\varepsilon_2}{\mu_2}}\left(E_{t\parallel}\,\hat{k} + E_{t\perp}\,(\operatorname{sen}\theta_t\,\hat{\imath} - \cos\theta_t\,\hat{j})\right)$$

Introduciendo en las condiciones de borde:

(1): $-E_{o\perp}\,\hat{j} + E_{o\parallel}\cos\theta\,\hat{k} - E_{r\perp}\,\hat{j} + E_{r\parallel}\cos\theta\,\hat{k}$

$= -E_{t\perp}\,\hat{j} + E_{t\parallel}\cos\theta_t\,\hat{k}$

luego

i) $E_{t\perp} - E_{r\perp} = E_{o\perp}$

ii) $E_{t\parallel}\cos\theta_t - E_{r\parallel}\cos\theta = E_{o\parallel}\cos\theta$

(2): $\sqrt{\frac{\varepsilon_1}{\mu_1}}\left[-E_{o\parallel}\,\hat{j} - E_{o\perp}\cos\theta\,\hat{k}\right] + \sqrt{\frac{\varepsilon_1}{\mu_1}}\left[E_{r\parallel}\,\hat{j} + E_{r\perp}\cos\theta\,\hat{k}\right]$

$= \sqrt{\frac{\varepsilon_2}{\mu_2}}\left[-E_{t\parallel}\,\hat{j} - E_{t\perp}\cos\theta_t\,\hat{k}\right]$

luego:

iii) $\sqrt{\varepsilon_2/\mu_2}\,\cos\theta_t\,E_{t\perp} + \sqrt{\varepsilon_1/\mu_1}\,\cos\theta\,E_{r\perp} = \sqrt{\varepsilon_1/\mu_1}\,\cos\theta\,E_{o\perp}$

iv) $\sqrt{\varepsilon_2/\mu_2}\,E_{t\parallel} + \sqrt{\varepsilon_1/\mu_1}\,E_{r\parallel} = \sqrt{\varepsilon_1/\mu_1}\,E_{o\parallel}$

cuatro ecuaciones y cuatro incógnitas

Componentes $\perp$: (4)

$$E_t\perp - E_r\perp = E_o\perp$$

$$\sqrt{\varepsilon_2/\mu_2}\cos\theta_t\, E_t\perp + \sqrt{\varepsilon_1/\mu_1}\cos\theta\, E_r\perp = \sqrt{\varepsilon_1/\mu_1}\cos\theta\, E_o\perp$$

$$E_t\perp = \frac{2\sqrt{\varepsilon_1/\mu_1}\cos\theta\, E_o\perp}{\sqrt{\varepsilon_1/\mu_1}\cos\theta + \sqrt{\varepsilon_2/\mu_2}\cos\theta_t}$$

$$E_r\perp = \frac{\left(\sqrt{\varepsilon_1/\mu_1}\cos\theta - \sqrt{\varepsilon_2/\mu_2}\cos\theta_t\right) E_o\perp}{\sqrt{\varepsilon_1/\mu_1}\cos\theta + \sqrt{\varepsilon_2/\mu_2}\cos\theta_t}$$

Componente $\parallel$:

$$E_t\parallel \cos\theta_t - E_r\parallel\cos\theta = E_o\parallel\cos\theta$$

$$\sqrt{\varepsilon_2/\mu_2}\, E_t\parallel + \sqrt{\varepsilon_1/\mu_1}\, E_r\parallel = \sqrt{\varepsilon_1/\mu_1}\, E_o\parallel$$

$$E_t\parallel = \frac{2\sqrt{\varepsilon_1/\mu_1}\cos\theta\, E_o\parallel}{\sqrt{\varepsilon_1/\mu_1}\cos\theta_t + \sqrt{\varepsilon_2/\mu_2}\cos\theta}$$

$$E_r\parallel = \frac{\left(\sqrt{\varepsilon_1/\mu_1}\cos\theta_t - \sqrt{\varepsilon_2/\mu_2}\cos\theta\right) E_o\parallel}{\sqrt{\varepsilon_1/\mu_1}\cos\theta_t + \sqrt{\varepsilon_2/\mu_2}\cos\theta} \quad //$$

finalmente se puede demostrar que las ecuaciones (3) y (4) se encuentran incluidas en (1) y (2):

$$(3):\ -\varepsilon_1 E_o\parallel\sin\theta + \varepsilon_1 E_r\parallel\sin\theta = -\varepsilon_2 E_t\parallel\sin\theta_t \quad (D_{n1} = D_{n2})$$

$$\Leftrightarrow\ E_t\parallel + \frac{\varepsilon_1\sin\theta\, E_r\parallel}{\varepsilon_2\sin\theta_t} = \frac{\varepsilon_1\sin\theta}{\varepsilon_2\sin\theta_t} E_o\parallel$$

$$\Leftrightarrow\ E_t\parallel + \frac{\varepsilon_1}{\varepsilon_2}\frac{\sqrt{\varepsilon_2\mu_2}}{\sqrt{\varepsilon_1\mu_1}} E_r\parallel = \frac{\varepsilon_1}{\varepsilon_2}\frac{\sqrt{\varepsilon_2\mu_2}}{\sqrt{\varepsilon_1\mu_1}} E_o\parallel$$

$$\Leftrightarrow \sqrt{\varepsilon_2/\mu_2}\, E_{t\parallel} + \sqrt{\varepsilon_1/\mu_1}\, E_{r,\parallel} = \sqrt{\varepsilon_1/\mu_1}\, E_{0\parallel} \Leftrightarrow (iv)_{//}$$

$$(4): \sqrt{\varepsilon_1 \mu_1}\, E_{0\perp} \operatorname{sen}\theta + \sqrt{\varepsilon_1 \mu_1}\, E_{r\perp} \operatorname{sen}\theta = \sqrt{\varepsilon_2 \mu_2}\, E_{t\perp} \operatorname{sen}\theta_t \quad (B_{n1} = B_{n2})$$

$$\Leftrightarrow E_{t\perp} - \frac{\sqrt{\varepsilon_1 \mu_1}\operatorname{sen}\theta}{\sqrt{\varepsilon_2 \mu_2}\operatorname{sen}\theta_t} E_{r\perp} = \frac{\sqrt{\varepsilon_1 \mu_1}\operatorname{sen}\theta}{\sqrt{\varepsilon_2 \mu_2}\operatorname{sen}\theta_t} E_{0\perp}$$

$$\Leftrightarrow E_{t\perp} - E_{r\perp} = E_{0\perp} \Leftrightarrow (i)_{//}$$