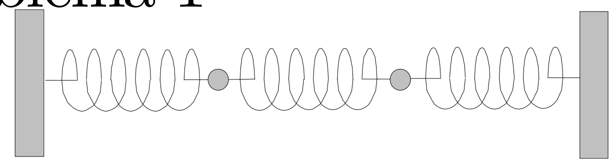
Problema 1



Si x_1 y x_2 indican las desviaciones de las partículas respecto a sus posiciones de equilibrio, la energía cinética es

$$K = \frac{1}{2}(m\dot{x}_1^2 + m\dot{x}_2^2),$$

y la energía potencial es

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}k_1(x_2 - x_1)^2 + \frac{1}{2}kx_2^2.$$

De allí las matrices $\mathbf{K}$ y $\mathbf{V}$ son

$$\begin{aligned} \mathbf{K} &= m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \mathbf{V} &= \begin{pmatrix} k+k_1 & -k_1 \\ -k_1 & k+k_1 \end{pmatrix}, \end{aligned}$$

por lo cual los valores propios satisfacen

$$\det\left(\omega^2 m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} k+k_1 & -k_1 \\ -k_1 & k+k_1 \end{pmatrix}\right) = 0,$$

o bien

$$\omega^4 m^2 - 2\omega^2 mk - 2\omega^2 mk_1 + k^2 + 2kk_1 = 0,$$

de donde resultan

$$\omega_1 = \sqrt{\frac{k}{m}}, \qquad \omega_2 = \sqrt{\frac{k+2k_1}{m}}.$$

El sistema de ecuaciones lineales es

$$\omega^2 m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} k+k_1 & -k_1 \\ -k_1 & k+k_1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

de donde se obtiene

$$\begin{aligned} (\omega^2 - \frac{(k+k_1)}{m})a_1 &= -\frac{k_1}{m}a_2, \\ a_1 &= -\frac{\frac{k_1}{m}}{(\omega^2 - \frac{(k+k_1)}{m})}a_2, \end{aligned}$$

y para las dos frecuencias propias resulta

$$\begin{aligned} a_{11} &= -\frac{\frac{k_1}{m}}{(\frac{k}{m} - \frac{(k+k_1)}{m})} a_{21} = a_{21}, \\ a_{12} &= -\frac{\frac{k_1}{m}}{(\frac{k+2k_1}{m} - \frac{(k+k_1)}{m})} a_{22} = -a_{22}. \end{aligned}$$

Normalización requiere que $\mathbf{A}^T\mathbf{K}\mathbf{A} = \mathbf{I}$, por lo cual

$$m \begin{pmatrix} a_{11} & a_{11} \\ -a_{22} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & -a_{22} \\ a_{11} & a_{22} \end{pmatrix} = I,$$

entonces $2ma_{11}^2 = 1$ y $2ma_{22}^2 = 1$, obteniendo

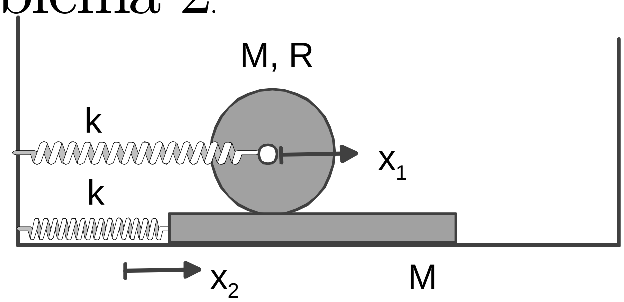
$$\mathbf{A} = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

También indicaremos los modos normales $\boldsymbol{\varsigma} = \mathbf{A}^T\mathbf{K}\boldsymbol{\eta}$ que resultan ser

$$\begin{aligned} \varsigma_1 &= \sqrt{\frac{m}{2}} \left(x_1 + x_2\right), \\ \varsigma_2 &= \sqrt{\frac{m}{2}} \left(-x_1 + x_2\right). \end{aligned}$$

El factor $\sqrt{\frac{m}{2}}$ es irrelevante.....

Problema 2.



Tenemos

$$\begin{aligned} \omega &= \frac{\dot{x}_1 - \dot{x}_2}{R} \\ K &= \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}\frac{1}{2}MR^2(\frac{\dot{x}_1 - \dot{x}_2}{R})^2 + \frac{1}{2}M\dot{x}_2^2 \\ &= \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}\frac{1}{2}M(\dot{x}_1 - \dot{x}_2)^2 + \frac{1}{2}M\dot{x}_2^2 \\ &= \frac{3}{4}M\dot{x}_1^2 - \frac{1}{2}M(\dot{x}_1\dot{x}_2) + \frac{3}{4}M\dot{x}_2^2 \\ V &= \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 \end{aligned}$$

De allí las matrices $\mathbf{K}$ y $\mathbf{V}$ son

$$\begin{aligned}\mathbf{K} &= M\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}, \\ \mathbf{V} &= \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix},\end{aligned}$$

por lo cual los valores propios satisfacen

$$\det\left(\omega^2 M\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix} - \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\right) = 0,$$

o bien

$$2\omega^4 M^2 - 3\omega^2 Mk + k^2 = 0,$$

de donde resulta

$$\omega_1^2 = \frac{k}{M}, \qquad \omega_2^2 = \frac{k}{2M}.$$

El sistema de ecuaciones lineales es

$$\omega^2 M\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

de donde se obtiene

$$a_1 = \frac{\frac{1}{2}\omega^2}{\frac{3}{2}\omega^2 - \frac{k}{M}} a_2,$$

y para las dos frecuencias propias resultan

$$\begin{aligned} a_{11} &= \frac{\frac{1}{2}\frac{k}{M}}{\frac{3}{2}\frac{k}{M} - \frac{k}{M}} a_{21} = a_{21}, \\ a_{12} &= \frac{\frac{1}{2}\frac{k}{2M}}{\frac{3}{2}\frac{k}{2M} - \frac{k}{M}} a_{22} = -a_{22}. \end{aligned}$$

Normalización requiere que $\mathbf{A}^T\mathbf{K}\mathbf{A} = \mathbf{I}$, por lo cual

$$\begin{aligned} M\begin{pmatrix} a_{11} & a_{11} \\ -a_{22} & a_{22} \end{pmatrix}\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}\begin{pmatrix} a_{11} & -a_{22} \\ a_{11} & a_{22} \end{pmatrix} &= I, \\ M\begin{pmatrix} 2a_{11}^2 & 0 \\ 0 & 4a_{22}^2 \end{pmatrix} &= I \end{aligned}$$

entonces $2Ma_{11}^2 = 1$ y $4Ma_{22}^2 = 1$, obteniendo

$$\mathbf{A} = \frac{1}{\sqrt{2M}}\begin{pmatrix} 1 & -\frac{1}{\sqrt{2}} \\ 1 & \frac{1}{\sqrt{2}} \end{pmatrix},$$

También indicaremos los modos normales $\boldsymbol{\varsigma} = \mathbf{A}^T\mathbf{K}\boldsymbol{\eta}$ que resultan ser

$$
\begin{aligned}
\mathbf{A}^T\mathbf{K} &= \frac{M}{\sqrt{2M}}\begin{pmatrix} 1 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix} \\
&= \frac{M}{\sqrt{2M}}\begin{pmatrix} 1 & 1 \\ -\sqrt{2} & \sqrt{2} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\varsigma_1 &= \sqrt{\frac{M}{2}}\,(x_1+x_2)\,, \\
\varsigma_2 &= \sqrt{M}\,(-x_1+x_2)\,.
\end{aligned}
$$

El factor $\sqrt{M}$ es irrelevante......

Problema 3.

Tenemos condiciones iniciales

$$
y(x,0)=0; \qquad \left.\frac{\partial y}{\partial t}\right|_{t=0} = 5\sin\frac{3\pi x}{L} - 2\sin\frac{5\pi x}{L}
$$

Podemos usar directamente D'Alembert

$$
\begin{aligned}
y(x,t) &= \frac{1}{2v}\int_{x-vt}^{x+vt}(5\sin\frac{3\pi u}{L} - 2\sin\frac{5\pi u}{L})du \\
&= \frac{1}{2v}\left.(-\frac{L}{3\pi}5\cos\frac{3\pi u}{L} + \frac{L}{5\pi}2\cos\frac{5\pi u}{L})\right|_{x-vt}^{x+vt} \\
&= \frac{L}{2\pi v}\left(-\frac{5}{3}(\cos\frac{3\pi(x+vt)}{L} - \cos\frac{3\pi(x-vt)}{L}) + \frac{2}{5}(\cos\frac{5\pi(x+vt)}{L} - \cos\frac{5\pi(x-vt)}{L})\right) \\
&= \frac{L}{2\pi v}\left(-\frac{5}{3}(\cos\frac{3\pi(x+vt)}{L} - \cos\frac{3\pi(x-vt)}{L}) + \frac{2}{5}(\cos\frac{5\pi(x+vt)}{L} - \cos\frac{5\pi(x-vt)}{L})\right) \\
&= \frac{5L}{3\pi v}\sin\frac{3\pi vt}{L}\sin\frac{3\pi x}{L} - \frac{2L}{5\pi v}\sin\frac{5\pi x}{L}\sin\frac{5\pi v}{L}
\end{aligned}
$$

O series de Fourier

$$
\begin{aligned}
y(x,t) &= \sum_{n=1}^{\infty} B_n \sin\frac{n\pi vt}{L}\sin\frac{n\pi x}{L} \\
B_n &= \frac{2}{n\pi v}\int_0^L (5\sin\frac{3\pi x}{L} - 2\sin\frac{5\pi x}{L})\sin\frac{n\pi x}{L}dx
\end{aligned}
$$

resultan distintos de cero B_3 y B_5

$$
B_3 = \frac{5L}{3\pi v}, \quad B_5 = -\frac{2L}{5\pi v}
$$

luego

$$
y(x,t) = \frac{5L}{3\pi v}\sin\frac{3\pi vt}{L}\sin\frac{3\pi x}{L} - \frac{2L}{5\pi v}\sin\frac{5\pi vt}{L}\sin\frac{5\pi x}{L}
$$