Problema 1

Calculamos las constantes

$$\begin{aligned}2E - Cs^2 &= A\dot{\phi}^2 \sin^2\theta + A\dot{\theta}^2 + 2mgh\cos\theta = 0\\ \alpha &= A\dot{\phi}\sin^2\theta + Cs\cos\theta = 0\end{aligned}$$

luego $f(u)$ será

$$\begin{aligned}\dot{u}^2 &= f(u) = \left(2E - Cs^2 - 2mghu\right)\frac{1-u^2}{A} - \left(\frac{\alpha - Csu}{A}\right)^2,\\ &= \left(-2mghu\right)\frac{1-u^2}{A} - \left(\frac{Csu}{A}\right)^2 = 0\end{aligned}$$

Pero $C = mh^2$, $A = 2mh^2$, $s = \sqrt{\frac{g}{h}}$ se obtiene

$$\begin{aligned}u(-4 + 4u^2 - u) &= 0\\ u_1 &= 0 \Rightarrow \theta_1 = \pi/2 = 90^o\\ u_2 &= \frac{1}{8} - \frac{1}{8}\sqrt{65} = -0.882\,8 \Rightarrow \theta_2 = 151.\,979\,8^o\end{aligned}$$

además

$$\dot{\phi} = -\frac{Cs\cos\theta}{A\sin^2\theta} = \frac{1}{2}\sqrt{\frac{g}{h}}\frac{\cos\theta}{-1+\cos^2\theta}$$

Nota: $f(u) = \frac{u}{-1+u^2}$, $f'(u) = -\frac{1+u^2}{(-1+u^2)^2} < 0$, el máximo ocurre en $u_2 = \frac{1}{8} - \frac{1}{8}\sqrt{65}$

$$\dot{\phi}_{\max} = \frac{1}{2}\sqrt{\frac{g}{h}}\frac{(\frac{1}{8} - \frac{1}{8}\sqrt{65})}{-1 + (\frac{1}{8} - \frac{1}{8}\sqrt{65})^2} = 2\sqrt{\frac{g}{h}}$$

Problema 2

. Sean x, y las coordenadas del centro de masa

$$\begin{aligned}x &= R\theta - h\sin\theta,\\ y &= R - h\cos\theta,\end{aligned}$$

luego

$$\begin{aligned}\dot{x} &= R\dot{\theta} - h\dot{\theta}\cos\theta,\\ \dot{y} &= h\dot{\theta}\sin\theta,\\ v_G^2 &= R^2\dot{\theta}^2 - 2Rh\dot{\theta}^2\cos\theta + h^2\dot{\theta}^2,\end{aligned}$$

luego el lagrangeano será

$$L = \frac{1}{2}M(R^2 - 2Rh\cos\theta + h^2)\dot{\theta}^2 + \frac{1}{2}I_g\dot{\theta}^2 - Mg(R - h\cos\theta)$$

hacemos las derivadas

$$\begin{aligned}\frac{\partial L}{\partial \dot{\theta}} &= (MR^2 - 2MRh\cos\theta + Mh^2 + I_G)\dot{\theta} \\ \frac{\partial L}{\partial \theta} &= M(Rh\sin\theta)\dot{\theta}^2 - Mgh\sin\theta\end{aligned}$$

finalmente

$$(MR^2 - 2MRh\cos\theta + Mh^2 + I_G)\ddot{\theta} + (MRh\sin\theta)\dot{\theta}^2 + Mgh\sin\theta = 0$$

Problema 3

. Las matrices de rotación construidas para $\phi = \pi/2$ resultan

$$R_{\hat{n}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix} + \begin{pmatrix} -n_y^2 - n_z^2 & n_x n_y & n_x n_z \\ n_x n_y & -n_x^2 - n_z^2 & n_y n_z \\ n_x n_z & n_y n_z & -n_x^2 - n_y^2 \end{pmatrix}$$

para los ejes

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

:

$$R_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$R_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

la rotación equivalente será

$$R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

de donde la traza es

$$Tr(R) = 1 = 1 + 2\cos\phi \Rightarrow \phi = \frac{\pi}{2}$$

además

$$\begin{aligned} R - R^T &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \\ &= \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix}\end{aligned}$$

de donde el eje es

$$\hat{n} = (0,1,0) = \hat{\jmath}$$

Problema 4

a) Usando

$$\frac{d\vec{L}_A}{dt} = \vec{\Gamma}_A - M\overrightarrow{AG} \times \vec{a}_A$$

siendo

$$\begin{aligned}
\vec{L}_A &= I_A\dot{\theta}\hat{k} = \frac{1}{3}ML^2\dot{\theta}\hat{k} \\
\vec{\Gamma}_A &= -Mg\frac{L}{2}\sin\theta\hat{k} \\
a_A &= R\omega^2 \text{ hacia el centro}
\end{aligned}$$

luego

$$\overrightarrow{AG} \times \vec{a}_A = -\frac{L}{2}R\omega^2\sin(90 - \omega t + \theta)\hat{k}$$

colocando todo junto

$$\begin{aligned}
\frac{1}{3}ML^2\ddot{\theta} &= -Mg\frac{L}{2}\sin\theta + M\frac{L}{2}R\omega^2\sin(90 - \omega t + \theta) \\
\ddot{\theta} &= -\frac{3g}{2L}\sin\theta + \frac{3}{2L}R\omega^2\cos(\omega t - \theta).
\end{aligned}$$

b) Por Lagrange

$$\begin{aligned}
x_G &= R\cos\omega t + \frac{L}{2}\sin\theta \\
y_G &= R\sin\omega t - \frac{L}{2}\cos\theta
\end{aligned}$$

$$\begin{aligned}
\dot{x}_G &= -\omega R\sin\omega t + \frac{L}{2}\dot{\theta}\cos\theta \\
y_G &= \omega R\cos\omega t + \frac{L}{2}\dot{\theta}\sin\theta
\end{aligned}$$

luego

$$\begin{aligned}
L &= \frac{1}{2}M(\omega^2R^2 - \omega RL\dot{\theta}\sin\omega t\cos\theta + \omega RL\dot{\theta}\cos\omega t\sin\theta + \frac{L^2}{4}\dot{\theta}^2) + \cdots \\
&\quad + \frac{1}{2}\frac{1}{12}ML^2\dot{\theta}^2 - Mg(R\sin\omega t - \frac{L}{2}\cos\theta)
\end{aligned}$$

podemos eliminar términos sin variables o velocidades obteniendo

$$L = \frac{1}{6}ML^2\dot{\theta}^2 - \frac{1}{2}M\omega RL\dot{\theta}\sin(\omega t - \theta) + Mg\frac{L}{2}\cos\theta$$

de aquí:

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}} &= \frac{1}{3}ML^2\dot{\theta} - \frac{1}{2}M\omega RL\sin(\omega t - \theta) \\
\frac{\partial L}{\partial \theta} &= \frac{1}{2}M\omega RL\dot{\theta}\cos(\omega t - \theta) - Mg\frac{L}{2}\sin\theta
\end{aligned}$$

luego

$$\frac{1}{3}ML^2\ddot{\theta} - \frac{1}{2}M\omega RL(\omega - \dot{\theta})\cos(\omega t - \theta) - \frac{1}{2}M\omega RL\dot{\theta}\cos(\omega t - \theta) + Mg\frac{L}{2}\sin\theta = 0$$

luego

$$\ddot{\theta} = -\frac{3g}{2L}\sin\theta + \frac{3}{2L}\omega^2 R\cos(\omega t - \theta)$$

igual por ambos métodos....