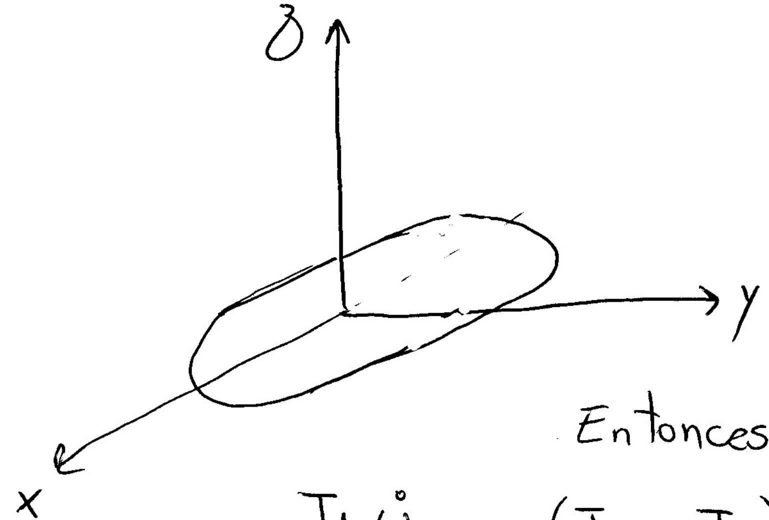
Ejercicio 3.6.3



Euler $\Rightarrow$ elección de un sistema no inercial solidario al sólido; los ejes principales $\Rightarrow$ matriz de Inercia diagonal.

Entonces:

$$I_1\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\omega_3 = \cancelto{0}{\tau_1^{ext}}$$
$$I_2\dot{\omega}_2 - (I_3 - I_1)\,\omega_3\omega_1 = \cancelto{0}{\tau_2^{ext}}$$
$$I_3\dot{\omega}_3 - (I_1 - I_2)\,\omega_1\omega_2 = \cancelto{0}{\tau_3^{ext}}$$

y $I_1 = A$; $I_2 = B$; $I_3 = C$

$1 \to x$
$2 \to y$
$3 \to z$

Debido a que se trata una placa: $dm = \rho_{superficial} \cdot dA = \frac{M}{\pi ab} \cdot dxdy$

$$\Rightarrow I_1 = \int y^2\,dm \ ; \ I_2 = \int x^2\,dm \ ; \ I_3 = \int (y^2 + x^2)\,dm$$

$$\Rightarrow I_3 = I_1 + I_2 \Rightarrow \boxed{C = A + B}$$

Entonces:

$$A\dot{\omega}_x - (\cancel{B} - A - \cancel{B})\,\omega_2\omega_3 = 0 \Rightarrow A\dot{\omega}_x + A\omega_y\omega_z = 0 \qquad (1)$$
$$B\dot{\omega}_y - (\cancel{A} + B - \cancel{A})\,\omega_3\omega_1 = 0 \Rightarrow B\dot{\omega}_y - B\omega_z\omega_x = 0 \qquad (2)$$
$$(A+B)\dot{\omega}_z - (A-B)\,\omega_1\omega_2 = 0 \Rightarrow \dot{\omega}_z = \frac{(A-B)}{(A+B)}\omega_x\omega_y \qquad (3)$$

$(1) \Rightarrow \quad \dot{\omega}_x + \omega_y \omega_z = 0 \quad (1.a)$

$(2) \Rightarrow \quad \dot{\omega}_y - \omega_z \omega_x = 0 \quad (2.a)$

$$(2.a)\cdot \omega_x + (2.b)\cdot \omega_y \Rightarrow \dot{\omega}_x \omega_x + \dot{\omega}_y \omega_y = 0$$

$$\Rightarrow \frac{d}{dt}(\omega_x^2 + \omega_y^2) = 0$$

$$\Rightarrow \boxed{\omega_x^2 + \omega_y^2 = cte}$$

pero en $t=0 \Rightarrow \vec{\omega}(t=0) = \omega \cos\left(\frac{\pi}{4}\right)\hat{x} + \omega \operatorname{sen}\left(\frac{\pi}{4}\right)\hat{y}$

$$= n \cos\left(\frac{\pi}{4}\right)\hat{x} + n \operatorname{sen}\left(\frac{\pi}{4}\right)\hat{y}$$

$$\Rightarrow \underbrace{\begin{array}{l} \omega_x(t=0) = n\cos\left(\frac{\pi}{4}\right) \\ \omega_y(t=0) = n\operatorname{sen}\left(\frac{\pi}{4}\right) \end{array}} \Rightarrow \omega_x^{0\,2} + \omega_y^{0\,2} = n^2$$

$$\Rightarrow \boxed{\omega_x^2 + \omega_y^2 = n^2} \qquad \Rightarrow \left.\begin{array}{l} \omega_x(t) = n\cos\left(\frac{\phi}{2}\right) \\ \omega_y(t) = n\operatorname{sen}\left(\frac{\phi}{2}\right) \end{array}\right\} (4)$$

(4) en (1.a) $\Rightarrow -\cancel{n}\frac{\dot{\phi}}{2}\cancel{\operatorname{sen}\left(\frac{\phi}{2}\right)} + \cancel{n}\cancel{\operatorname{sen}\left(\frac{\phi}{2}\right)}\omega_z = 0$

$$\Rightarrow \boxed{\omega_z = \frac{\dot{\phi}}{2}} \quad (5)$$

(5) en (3)

$$\Rightarrow \frac{\ddot{\phi}}{2} = \frac{(A-B)}{(A+B)} n^2 \cos\left(\frac{\phi}{2}\right)\operatorname{sen}\left(\frac{\phi}{2}\right) = \frac{(A-B)}{(A+B)}\cdot\frac{1}{2}\cdot n^2 \cdot \operatorname{sen}(\phi)$$

$$\Rightarrow \boxed{\ddot{\phi} = n^2 \frac{(A-B)}{(A+B)} \operatorname{sen}\phi}$$

$$\frac{d\dot{\phi}}{dt} = \Omega^2 \operatorname{sen}\phi \quad ; \quad \Omega^2 = m^2 \frac{(A-B)}{(A+B)}$$

$$\Rightarrow \frac{d\dot{\phi}}{d\phi}\cdot\dot{\phi} = \Omega^2 \operatorname{sen}\phi$$

$$\Rightarrow \int_0^{\dot{\phi}} \dot{\phi}\, d\dot{\phi} = \int_{\pi/2}^{\phi} \Omega^2 \operatorname{sen}\phi \, d\phi$$

con: $\dot{\phi}(0) = 0$ $\left(\begin{array}{l}\text{ya que} \\ \dot{\phi} = 2\omega_z \\ \text{y en } t=0 \\ \Rightarrow \omega_z = 0\end{array}\right)$

$\phi(0) = \frac{\pi}{2}$ $\left(\begin{array}{l}\text{Cond. inicial} \\ \text{dada}\end{array}\right)$

$$\Rightarrow \frac{\dot{\phi}^2}{2} = \Omega^2 \Big[-\cos\phi \Big]_{\pi/2}^{\phi}$$

$$\Rightarrow \frac{\dot{\phi}^2}{2} = -\Omega^2 \cos\phi = W^2 \cos\phi$$

$$\Rightarrow \frac{d\phi}{dt} = \sqrt{-2\Omega^2 \cos\phi} \quad \left(\begin{array}{c}\text{atención !!} \\ \sqrt{-\text{algo}}\,?\end{array}\right)$$

$\Omega^2 = m^2 \frac{(A-B)}{(A+B)}$ pero $A = \frac{Mb^2}{4}$; $B = \frac{Ma^2}{4}$ con $a > b$

$$\Rightarrow \boxed{A - B < 0}$$

$$\Rightarrow \Omega^2 < 0$$

Renombramos:

$$-\Omega^2 = W^2 = m^2 \frac{(B-A)}{(A+B)}$$

$$\Rightarrow d\phi = W\sqrt{2}\sqrt{\cos\phi}\, dt$$

$$\Rightarrow \frac{d\phi}{\sqrt{\cos\phi}} = w\sqrt{2}\,dt \Rightarrow \int_0^T dt = \frac{1}{w\sqrt{2}}\int \frac{d\phi}{\sqrt{\cos\phi}}$$

lo que queremos es que ω_z vuelva a ser cero.

$$\Rightarrow \phi(T) = 0 \Rightarrow \cos\phi = 0 \Rightarrow \phi = \frac{\pi}{2}, \frac{3\pi}{2}$$

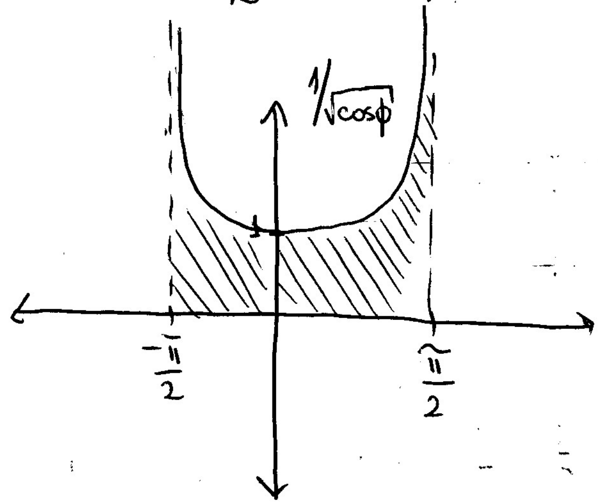
pero $\phi = \frac{\pi}{2}$ es $\phi(t=0)$

$$\Rightarrow \phi(T) = \frac{3\pi}{2}$$

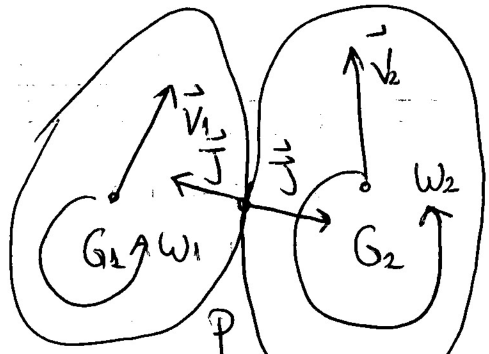
$$\Rightarrow T = \frac{1}{w\sqrt{2}}\int_{\pi/2}^{3\pi/2} \frac{d\phi}{\sqrt{\cos\phi}}$$

pero $\frac{1}{\sqrt{\cos\phi}} \in \mathbb{R} \;\; \forall \phi \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

$$\Rightarrow T = \frac{1}{w\sqrt{2}}\int_{\pi/2}^{-\pi/2} \frac{d\phi}{\sqrt{\cos\phi}} = \frac{2}{w\sqrt{2}}\int_0^{\pi/2} \frac{d\phi}{\sqrt{\cos\phi}}$$



Ejercicio 3.6.4



Sea $\vec{J}$ el impulso

En términos globales:

$$\vec{J} = \Delta\vec{P} = \vec{F}\cdot\Delta t$$

$\mathbb{I}_1$: matriz inercia cuerpo 1.

$\mathbb{I}_2$: matriz " " 2.

Entonces:

$$\vec{J} = M_1\left(\vec{V}_1^+ - \vec{V}_1^-\right) \quad (1)$$

$$\underbrace{-\vec{J} = M_2\left(\vec{V}_2^+ - \vec{V}_2^-\right) \quad (2)}_{\text{"momentum lineal"}}$$

$$\Delta\vec{L}_1 = \mathbb{I}_1\left(\vec{\omega}_1^+ - \vec{\omega}_1^-\right) = \vec{G_1P}\times\vec{J} \quad (3)$$

$$\underbrace{\Delta\vec{L}_2 = \mathbb{I}_2\left(\vec{\omega}_2^+ - \vec{\omega}_2^-\right)}_{\text{"momentum angular"}} = -\vec{G_2P}\times\vec{J} \quad (4)$$

Por energía: (choque elástico)

$$K^{\text{antes}}_{\text{traslación}} + K^{\text{antes}}_{\text{rotación}} = K^{\text{después}}_{\text{traslación}} + K^{\text{después}}_{\text{rotación}}$$

$$\frac{1}{2}M_1\vec{V}_1^{-2} + \frac{1}{2}M_2\vec{V}_2^{-2} + \frac{1}{2}\mathbb{I}_1\vec{\omega}_1^{-2} + \frac{1}{2}\mathbb{I}_2\vec{\omega}_2^{-2} = \frac{1}{2}M_1\vec{V}_1^{+2} + \frac{1}{2}M_2\vec{V}_2^{+2} + \frac{1}{2}\mathbb{I}_1\vec{\omega}_1^{+2}$$

$$+\frac{1}{2}\mathbb{I}_2\vec{\omega}_2^{+2} \quad /\cdot 2 \quad (*)$$

pero: $\vec{\omega}_1^+(3) + \vec{\omega}_1^-(3) \Rightarrow$

$$\vec{\omega}_1^+\mathbb{I}_1\vec{\omega}_1^+ - \vec{\omega}_1^+\mathbb{I}_1\vec{\omega}_1^- = \vec{\omega}_1^+\cdot\left(\vec{G_1P}\times\vec{J}\right)$$

$$+\ \vec{\omega}_1^-\mathbb{I}_1\vec{\omega}_1^+ - \vec{\omega}_1^-\mathbb{I}_1\vec{\omega}_1^- = \vec{\omega}_1^-\left(\vec{G_1P}\times\vec{J}\right)$$

pero $\left(\vec{\omega}_1^+\mathbb{I}_1\vec{\omega}_1^-\right)^T = \vec{\omega}_1^-\mathbb{I}_1^T\vec{\omega}_1^+ = \vec{\omega}_1^-\mathbb{I}_1\vec{\omega}_1^+$

$$\Rightarrow \vec{\omega}_1^{+} \mathbb{I}_1 \vec{\omega}_1^{+} - \vec{\omega}_1^{-} \mathbb{I}_1 \vec{\omega}_1^{-} = (\vec{\omega}_1^{+} + \vec{\omega}_1^{-})(\vec{G_1P} \times \vec{J}) \quad (5)$$

Análogamente:

$$\vec{\omega}_2^{+} \mathbb{I}_2 \vec{\omega}_2^{+} - \vec{\omega}_2^{-} \mathbb{I}_2 \vec{\omega}_2^{-} = -(\vec{\omega}_2^{+} + \vec{\omega}_2^{-})(\vec{G_2P} \times \vec{J}) \quad (6)$$

pero de $(*)$

$$M_1 \underbrace{(\vec{V}_1^{+2} - \vec{V}_1^{-2})}_{(\vec{V}_1^{+} + \vec{V}_1^{-})(\vec{V}_1^{+} - \vec{V}_1^{-})} + M_2(\vec{V}_2^{+2} - \vec{V}_2^{-2}) = \mathbb{I}_1(\vec{\omega}_1^{-2} - \vec{\omega}_1^{+2}) + \mathbb{I}_2(\vec{\omega}_2^{-2} - \vec{\omega}_2^{+2})$$

$$\Rightarrow (\vec{V}_1^{+} + \vec{V}_1^{-}) \cdot \vec{J} - (\vec{V}_2^{+} + \vec{V}_2^{-}) \cdot \vec{J} = \mathbb{I}_1(\vec{\omega}_1^{-} - \vec{\omega}_1^{+})(\vec{\omega}_1^{-} + \vec{\omega}_1^{+}) + \mathbb{I}_2(\vec{\omega}_2^{-} - \vec{\omega}_2^{+})(\vec{\omega}_2^{-} + \vec{\omega}_2^{+})$$

$$= -(\vec{\omega}_1^{-} + \vec{\omega}_1^{+})(\vec{G_1P} \times \vec{J}) + (\vec{G_2P} \times \vec{J})(\vec{\omega}_2^{-} + \vec{\omega}_2^{+})$$

$$= -(\vec{\omega}_1^{-} + \vec{\omega}_1^{+}) \times \vec{G_1P} \cdot \vec{J} + (\vec{\omega}_2^{-} + \vec{\omega}_2^{+}) \times \vec{G_2P} \cdot \vec{J}$$

pero: $\vec{V}_{P_1}^{+} = \vec{V}_1^{+} + \vec{\omega}_1^{+} \times \vec{G_1P}$

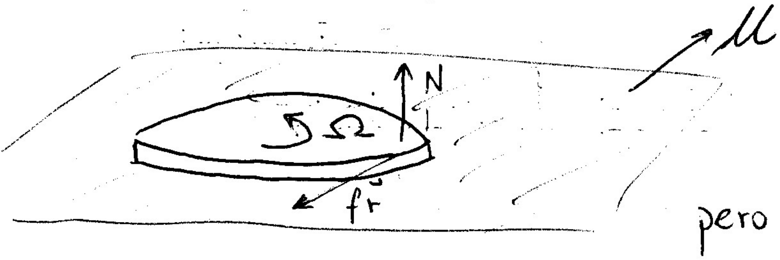
$$\Rightarrow (\vec{V}_1^{+} + \vec{V}_1^{-}) \cdot \vec{J} + (\vec{\omega}_1^{-} + \vec{\omega}_1^{+}) \times \vec{G_1P} \cdot \vec{J} = (\vec{V}_2^{+} + \vec{V}_2^{-}) \cdot \vec{J} + (\vec{\omega}_2^{-} + \vec{\omega}_2^{+}) \times \vec{G_2P} \cdot \vec{J}$$

$$(\vec{V}_{p_1}^{+} + \vec{V}_{p_1}^{-}) \cdot \vec{J} = (\vec{V}_{p_2}^{+} + \vec{V}_{p_2}^{-}) \cdot \vec{J}$$

$$\Rightarrow (\vec{V}_{p_1}^{+} - \vec{V}_{p_2}^{+}) \cdot \vec{J} = -(\vec{V}_{p_1}^{-} - \vec{V}_{p_2}^{-}) \cdot \vec{J}$$

$$\Rightarrow \boxed{\vec{V}_{+} \cdot \hat{J} = -\vec{V}_{-} \hat{J}}$$

Ejercicio 3.6.6



$$dN = dm\,g \quad (\text{por qué?})$$

pero $dm = \dfrac{M}{\pi a^2} \cdot r\,dr\,d\theta$

si consideramos discos de distintos radios tendríamos distintas masas por lo que la fuerza sería distinta para cada disco

$$\Rightarrow \boxed{dN = \frac{Mg\,r\,dr\,d\theta}{\pi a^2}}$$

$$\Rightarrow \text{"}d\vec{f}_r\text{"} = -\mu \cdot \frac{Mg\,r\,dr\,d\theta}{\pi a^2} \cdot \hat{\theta}$$

$$\Rightarrow \text{"}d\vec{\tau}_0^{\,fr}\text{"} = -\mu \frac{Mg\,r^2\,dr\,d\theta}{\pi a^2} \underbrace{(\hat{r} \times \hat{\theta})}_{\hat{k}}$$

$$\Rightarrow \vec{\tau}_0^{\,fr} = -\mu \frac{Mg}{\cancel{\pi}\cancel{a^2}} \frac{a^{\cancel{3}}}{3} 2\cancel{\pi}\,\hat{k} = -\frac{2}{3}\mu M g a\,\hat{k}$$

pero $\vec{\tau}_0 = I_1 \cdot \alpha\,\hat{k}$

$$\Rightarrow I \cdot \alpha = -\frac{2}{3}\mu M g a$$

pero $I = \dfrac{1}{2} M a^2$

$$\Rightarrow \alpha = -\frac{2}{3}\mu \cancel{M} g \cancel{a} \cdot \frac{2}{\cancel{M} a^{\cancel{2}}} = -\frac{4}{3}\frac{\mu g}{a}$$

$$\Rightarrow \boxed{\alpha = -\frac{4}{3}\frac{\mu g}{a}}$$

el cuerpo des-acelera.
con aceleración cte.

$$\Rightarrow \quad v_f = v_i + a \cdot t$$

$$\Rightarrow \quad \omega_f = \omega_i + \alpha \cdot t$$

$$\Rightarrow \quad 0 = \Omega - \frac{4}{3}\frac{\mu g}{a} \cdot t$$

se obtiene $\quad \Rightarrow \quad \boxed{t = \frac{3a\Omega}{4\mu g}}$