Pregunta 2 - Control 3

MA26A

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Resolver $\dot{X}(t) = AX(t) + B(t)$ con:

$$A = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & -3 & 4 \end{bmatrix} ; B = \begin{pmatrix} 0 \\ 0 \\ sen(2t) \\ 0 \\ 0 \end{pmatrix} ; X_0 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

Solución

(a)

- Como A es diagonal por bloques: $e^{At} = \begin{bmatrix} e^{M_1 t} & 0 \\ 0 & e^{M_2 t} \end{bmatrix}$

$$M_1 = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & -2 \end{bmatrix} \Rightarrow det(M_1 - \lambda I) = \begin{vmatrix} 4-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 1 & 1 & -2-\lambda \end{vmatrix}$$

$$\Rightarrow -(4-\lambda)(3-\lambda)(2+\lambda) = 0 \Rightarrow \begin{array}{c} \lambda_1 = 4 \\ \lambda_2 = 3 \\ \lambda_3 = -2 \end{array}$$

(0.5 ptos)

- Vectores propios
  $\lambda = 4$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & -6 \end{bmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{c} C_1 = 0 \\ C_2 = 6C_3 \\ C_3 = libre \end{array} \Rightarrow V_{p_{\lambda=4}} = \begin{pmatrix} 6 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda = 3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & -5 \end{bmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{c} C_1 = -C_2 \\ C_3 = 0 \end{array} \Rightarrow V_{p_{\lambda=3}} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda = -2$

$$\begin{bmatrix} 6 & 1 & 0 \\ 0 & 5 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} C_1 = 0 \\ C_2 = 0 \\ C_3 = libre \end{matrix} \Rightarrow V_{p_{\lambda=-2}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(1 pto)
Por lo tanto tenemos:

$$P = \begin{bmatrix} -1 & 6 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}; D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}; P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{6} & 0 \\ -\frac{1}{6} & -\frac{1}{6} & 1 \end{bmatrix}$$

Así tenemos:

$$e^{M_1 t} = \begin{bmatrix} -1 & 6 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{4t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{6} & 0 \\ -\frac{1}{6} & -\frac{1}{6} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{4t} & e^{4t} - e^{3t} & 0 \\ 0 & e^{3t} & 0 \\ \frac{e^{4t} - e^{-2t}}{6} & \frac{e^{4t} - e^{-2t}}{6} & e^{-2t} \end{bmatrix}$$

(0.5 pto)

$M_2 = \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$ es de la forma $\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$

$$\Rightarrow e^{M_2 t} = e^{4t} \begin{bmatrix} cos(3t) & sen(3t) \\ -sen(3t) & cos(3t) \end{bmatrix}$$

(1 pto)
Finalmente,

$$e^{At} = \begin{bmatrix} e^{4t} & e^{4t} - e^{3t} & 0 & 0 & 0 \\ 0 & e^{3t} & 0 & 0 & 0 \\ \frac{e^{4t} - e^{-2t}}{6} & \frac{e^{4t} - e^{-2t}}{6} & e^{-2t} & 0 & 0 \\ 0 & 0 & 0 & e^{4t} cos(3t) & e^{4t} sen(3t) \\ 0 & 0 & 0 & -e^{4t} sen(3t) & e^{4t} cos(3t) \end{bmatrix}$$

- Solución homogenea:

$$X_h = e^{At} X_0$$

$$\Rightarrow X_h = \begin{bmatrix} e^{4t} & e^{4t}-e^{3t} & 0 & 0 & 0 \\ 0 & e^{3t} & 0 & 0 & 0 \\ \frac{e^{4t}-e^{-2t}}{6} & \frac{e^{4t}-e^{-2t}}{6} & e^{-2t} & 0 & 0 \\ 0 & 0 & 0 & e^{4t}cos(3t) & e^{4t}sen(3t) \\ 0 & 0 & 0 & -e^{4t}sen(3t) & e^{4t}cos(3t) \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3e^{4t}-2e^{3t} \\ 2e^{3t} \\ \frac{e^{4t}}{2}+\frac{3e^{-2t}}{2} \\ e^{4t}cos(3t) \\ -e^{4t}sen(3t) \end{pmatrix}$$

(1 pto)

- Solución particular:

$$X_p = e^{At}\int_0^t e^{-As}B(s)ds$$

La inversa de e^{At} se obtiene cambiando t por $-t$

$$\Rightarrow e^{-As}B(s):$$

$$= \begin{bmatrix} e^{-4s} & e^{-4s}-e^{-3s} & 0 & 0 & 0 \\ 0 & e^{-3s} & 0 & 0 & 0 \\ \frac{e^{-4s}-e^{2s}}{6} & \frac{e^{-4s}-e^{2s}}{6} & e^{2s} & 0 & 0 \\ 0 & 0 & 0 & e^{-4s}cos(3s) & -e^{-4s}sen(3s) \\ 0 & 0 & 0 & e^{-4s}sen(3s) & e^{-4s}cos(3s) \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ sen(2s) \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ e^{2s}sen(2s) \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow X_p = e^{At}\begin{pmatrix} 0 \\ 0 \\ \frac{e^{2t}[sen(2t)-cos(2t)]+1}{4} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{sen(2t)-cos(2t)+e^{-2t}}{4} \\ 0 \\ 0 \end{pmatrix}$$

(1.5 pto)

Finalmente $X = X_p + X_h$ (0.5 ptos)