SECOND HOUR EXAM SUGGESTED SOLUTIONS

1. **Find the critical points** (if any), and **classify them as local maxima, minima or saddles**, for the function

$$f(x, y) = 2x^2y + 2x^2 + y^2 \ . \qquad [15 \text{ pts.}]$$

We have $\dfrac{\partial f}{\partial x} = 4xy + 4x$ *and* $\dfrac{\partial f}{\partial y} = 2x^2 + 2y$. *The coördinates of the critical points are the solutions of* $4xy + 4x = 0$ *or* $x(y + 1) = 0$ *and* $2x^2 + 2y = 0$ *or* $y = -x^2$, *so either* $x = 0 = y$ *or* $y = -1$ *and consequently* $x = \pm 1$. *Thus there are three critical points:* $(0, 0)$, $(1, -1)$ *and* $(-1, -1)$. *The discriminant of the problem is* $D = f_{xx}f_{yy} - f_{xy}^2 = 4(y + 1) \cdot 2 - (4x)^2 = -16x^2 + 8y + 8$. *Since* $D(0, 0) = 8 > 0$, $(0, 0)$ *is a point of definiteness, and because* $f_{yy} \equiv 2 > 0$ *it must be a local minimum. On the other hand,* $D(\pm 1, -1) = -16 < 0$ *so both these critical points are saddles.*

2. **Find the absolute minimum value** of the function $f(x, y) = 2x^2 + y^2 - y$ on the disc $x^2 + y^2 \le 1$, and also **find the coördinates of the point(s)** (x, y) **at which the minimum is attained**. This problem **must** be done in two parts: find the critical points in the interior of the disc where $x^2 + y^2 < 1$, and also find the minimum of $f(x, y)$ subject to the constraint $x^2 + y^2 = 1$. That may be done by the method of Lagrange multipliers, by parametrizing the unit circle (thus reducing the problem to a one-variable extremum problem), or by any other **correct** method at your disposal. [15 pts.]

The (unconstrained) critical points of f *are the solutions of the system* $4x = 0$ *and* $2y - 1 = 0$, *of which there is only one,* $(0, 1/2)$. *It lies in the interior of the disc, so it is a possible minimum point: the value* $f(0, 1/2) = -1/4$. *On the boundary of the disc where the constraint* $x^2 + y^2 = 1$ *holds one can seek extrema by the method of Lagrange multipliers. At an extremum one must have* $\nabla f = \lambda \nabla(x^2 + y^2)$, *which in component form is the system* $4x = \lambda 2x$, $2y - 1 = \lambda 2y$. *If* $x = 0$ *then* $y = \pm 1$ *from the constraint. If* $x \ne 0$ *then* $\lambda = 2$ *and therefore* $2y - 1 = 4y$ *or* $y = -1/2$; *the constraint then gives* $x = \pm\sqrt{3}/2$. *We have* $f(0, 1) = 0$ *and* $f(0, -1) = 2$, *while* $f(\pm\sqrt{3}/2, -1/2) = 9/4$. *It thus appears that the absolute minimum value is* $-1/4$, *occurring at* $(0, 1/2)$. *One can also approach the constrained problem by parametrizing the constraint curve, which is a circle of radius* 1 *centered at the origin, by (say)* $x = \cos t$, $y = \sin t$. *Then* $f(\cos t, \sin t) = 2\cos^2 t + \sin^2 t - \sin t = \cos^2 t + 1 - \sin t$ *with derivative* $-2\cos t \sin t - \cos t$. *The zeros of the derivative occur where* $x = \cos t = 0$ *and thus* $y = \sin t = \pm 1$, *or where* $-2\sin t - 1 = 0$ *or* $y = \sin t = -1/2$ *and thus* $x = \pm\sqrt{3}/2$; *the rest of the checking for extrema is the same as it was when Lagrange multipliers were used. Still another approach would plug the constraint into the objective function, which then becomes* $x^2 + 1 - y$. *Lagrange multipliers used on this function would lead to the equations* $2x = \lambda 2x$ *and* $-1 = \lambda 2y$, *so either* $x = 0$ *or* $\lambda = 1$ *and thus* $y = -1/2$, *and then things finish up as before.*(1)

3. **Sketch the region** in the xy-plane over which the integration

$$\int_0^1 \int_y^1 x^2 e^{xy} \, dx \, dy$$

takes place, and **write an iterated integral** of the form $\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} x^2 e^{xy} \, dy \, dx$ that represents the same double integral. **Evaluate the new integral (ONLY!).** [10 pts.]

The 2-dimensional region of integration is the lower or right-hand half of the unit square $[0, 1] \times [0, 1]$ *as bisected by the line* $y = x$. *Integrating over this region in the* y *direction first will lead to the integral*

$$\int_0^1 \int_0^x x^2 e^{xy} \, dy \, dx = \int_0^1 \left[\frac{x^2 e^{xy}}{x} \right]_{y=0}^{y=x} dx = \int_0^1 [e^{x^2} - 1] x \, dx = \left[\frac{e^{x^2} - x^2}{2} \right]_0^1 = \frac{e - 1}{2} - \frac{1}{2} = \frac{e}{2} - 1 \ .$$

(1) Note that when we rewrote the problem, the objective function changed and therefore so did the value of the Lagrange multiplier λ.

4. Consider the integral $\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} \frac{2}{\sqrt{4-x^2-y^2}}\, dx\, dy$. It is unpleasant to evaluate this integral in rectangular coördinates. **Sketch** the region of integration, **write an integral in polar coördinates** that represents the same double integral, and **evaluate the polar-coördinate integral**. {Note: the region is bounded by the x-axis and the circle of radius 1 centered at the origin, and lies above the x-axis. Use the chain rule for single-variable integrals carefully.} [15 pts.]

From the geometrical description of the region we see that in polar coördinates the limits of the integral are as below:

$$\int_0^{\pi/2} \int_0^1 \frac{2}{\sqrt{4-r^2}}\, r\, dr\, d\theta = \frac{\pi}{2} \cdot \left[-2\sqrt{4-r^2}\right]_0^1 = \pi \cdot (2-\sqrt{3}) \, .$$

5. **Set up, and then evaluate**, an iterated integral in 3-dimensional rectangular or cylindrical coördinates (your choice) whose value is the triple integral $\displaystyle\iiint_D z\, dV$, where D is the solid whose upper surface is the paraboloid $z = 12 - 2x^2 - 2y^2$ and whose lower surface is the paraboloid $z = x^2 + y^2$. **Make a sketch** of the part of the solid lying in the first octant (but be sure your integral includes all of the solid). [15 pts.]

The solid in question lies over the disc in the xy-plane bounded by the circle that lies under the circle in which the two paraboloids intersect. This is the solution set of $12 - 2x^2 - 2y^2 = x^2 + y^2$, or $4 = x^2 + y^2$, the circle of radius 2 centered at the origin. The symmetry of the solid about the z-axis tells us that cylindrical coördinates are to be preferred, so the iterated integral will have the form

$$\int_0^{2\pi} \int_0^2 \int_{r^2}^{12-2r^2} z\, dz\, r\, dr\, d\theta = 2\pi \int_0^2 \left[\frac{z^2}{2}\right]_{z=r^2}^{z=12-2r^2} r\, dr = \pi \int_0^2 \left[(12-2r^2)^2 - r^4\right] r\, dr$$

$$= \pi \int_0^2 (144r - 48r^3 + 3r^5)\, dr = \pi\left(72 \cdot 2^2 - 12 \cdot 2^4 + \frac{1}{2} \cdot 2^6 - 0\right) = 128\pi \, .$$

6. Consider the "spherical cap" cut by the cylinder $x^2 + y^2 = 1$ from the part of the sphere $x^2 + y^2 + z^2 = 2$ that lies above the xy-plane. **Sketch the spherical cap**. Then **find its surface area**. {Note: You may set up this problem in rectangular, cylindrical or spherical coördinates, but you will probably find that the resulting integral is **most** difficult in rectangular coördinates. You may therefore transform it, if necessary, to another coördinate system after setting it up. By the way, a correct answer will contain the constant $\sqrt{2}$.} [15 pts.]

If we set up the problem in rectangular coördinates first, the spherical cap will be part of the graph of $z = \sqrt{2 - x^2 - y^2}$. The first partials are $\dfrac{\partial z}{\partial x} = \dfrac{-x}{\sqrt{2-x^2-y^2}}$ and $\dfrac{\partial z}{\partial y} = \dfrac{-y}{\sqrt{2-x^2-y^2}}$, so the element of surface area is given by

$$dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA_{xy} = \sqrt{\frac{x^2 + y^2 + (2 - x^2 - y^2)}{2 - x^2 - y^2}}\, dA_{xy} = \frac{\sqrt{2}}{\sqrt{2-x^2-y^2}}\, dA_{xy} \, .$$

At this stage it is clear that the integration should be carried out in cylindrical coördinates (or polar coördinates in the plane, which comes to the same thing). The region of integration in the plane is the disc with radius 1 and center at the origin, so the integral takes the form

$$S = \int_0^{2\pi} \int_0^1 \frac{\sqrt{2}\, r\, dr\, d\theta}{\sqrt{2-r^2}} = 2\pi\sqrt{2}\left[-\sqrt{2-r^2}\right]_0^1 = 2\pi\sqrt{2}\left[\sqrt{2}-1\right] = 2\pi(2-\sqrt{2}) \, .$$

Actually, the easiest integrals occur if the problem is set up in spherical coördinates. One needs to remember (or derive) the fact that the element of area on a sphere of radius a is $dS = a^2 \sin\phi\, d\phi\, d\theta$. With that fact available, however, the integral takes the form

$$S = \int_0^{2\pi} \int_0^{\pi/4} 2\sin\phi\, d\phi\, d\theta = 4\pi\left[-\cos\phi\right]_0^{\pi/4} = 4\pi\left(1 - \frac{\sqrt{2}}{2}\right) = 2\pi(2-\sqrt{2}) \, .$$

The limit $\phi = \pi/4$ comes from the observation that a cross-section through the z-axis of the sphere and cylinder shows that the spherical cap subtends one of the angles of an isosceles right triangle (with hypotenuse $\sqrt{2}$ and side 1).

7. Evaluate the line integral $\int_C \mathbf{F} \bullet d\mathbf{r}$ where the vector field $\mathbf{F}(x, y) = \langle 2xy^3 + 1, 3x^2y^2 - 2y \rangle$ and the parametrized curve C is given by $\mathbf{r}(t) = \langle \sin t, 1 - \cos t \rangle$, where $0 \leq t \leq \pi$. {Note: You know at least two ways to approach this problem; both will give the same result and you may use whichever method you prefer.}

[15 pts.]

To run the "cross-partials equal" test we need $\dfrac{\partial(2xy^3+1)}{\partial y} = 6xy^2$ *to equal* $\dfrac{\partial(3x^2y^2-2y)}{\partial x} = 6xy^2$*, which it does. So* $\mathbf{F}$ *has a potential. A trial integral of its* $\mathbf{i}$*-component gives* $x^2y^3 + x$*, but* $\dfrac{\partial(x^2y^3+x)}{\partial y} = 3x^2y^2$ *lacks the term* $-2y$*; the potential* $f(x, y) = x^2y^3 + x - y^2$ *has* $\nabla f = \langle 2xy^3 + 1, 3x^2y^2 - 2y \rangle$ *as we want. The curve* C *begins at* $(0, 0)$ *and ends at* $(0, 2)$*, so* $\int_C \mathbf{F} \bullet d\mathbf{r} = f(0, 2) - f(0, 0) = -2 \cdot 2 - 0 = -4$*. Of course there is a hard way to do anything: on the curve one has* $\mathbf{F}(\sin t, 1 - \cos t) = \langle (2 \sin t)(1 - \cos t)^3 + 1, (3 \sin^2 t)(1 - \cos t)^2 - 2(1 - \cos t) \rangle$ *and* $\dfrac{d\mathbf{r}}{dt} = \langle \cos t, \sin t \rangle$*, so one can parametrize the curve, express everything in terms of the parameter* t*, and integrate over* $0 \leq t \leq \pi$ *to get*

$$\int_C \mathbf{F} \bullet d\mathbf{r} =$$

$$\int_0^{\pi} \left[\left(2 \sin(t) \left(1 - \cos(t)\right)^3 + 1\right) \cos(t) + \left(3 \left(\sin(t)\right)^2 \left(1 - \cos(t)\right)^2 - 2 + 2 \cos(t)\right) \sin(t)\right] dt =$$

$$\int_0^{\pi} \left[2 \sin(t) \cos(t) - 6 \sin(t) \left(\cos(t)\right)^2 + 12 \sin(t) \left(\cos(t)\right)^3 - 5 \sin(t) \left(\cos(t)\right)^4 + \cos(t) + \sin(t)\right] dt =$$

$$\left[\sin^2 t + 2\cos^3 t - 3\cos^4 t + \cos^5 t + \sin t - \cos t\right]_0^{\pi} = (-2 - 3 - 1 + 1) - (2 - 3 + 1 - 1) = -4\ ,$$

the same result.