A Proof that the Double Integral and Iterated Integral over a Rectangle are Equal

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Let R be a rectangle in the plane defined by $x \in [c, d], y \in [a, b]$. Suppose that $f(x, y)$ is a continuous function on R. We want to show that

$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dydx.$$

The proof will not depend on the order of integration in x and y, so reversing their roles in the subsequent proof shows that the integral is unchanged when integrating first with respect to x and then y.

We begin by fixing some notation. Let $(x_0, \ldots, x_m)$ be the endpoints of a partition of $[c, d]$ into m parts. So in particular, $x_0 = c$ and $x_m = d$. Similarly, let $(y_0, \ldots, y_n)$ be the endpoints of a partition of $[a, b]$. This gives a partition of the rectangle R into mn cells.

Roughly, we know that as this partition gets finer and finer, i.e. mn gets large, then

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\Delta y_j \; \Delta x_i \approx \iint_R f(x,y)dA$$

Precisely, we mean that since f is continuous, the limit of the Riemann sums exists:

$$\lim_{mn \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\Delta y_j \; \Delta x_i = \iint_R f(x,y)dA.$$

Recall that this just means that given any $\epsilon > 0$ there is an N such that if $mn > N$ then

$$\left| \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\Delta y_j \; \Delta x_i - \iint_R f(x,y)dA \right| < \epsilon. \tag{1}$$

Now we want to use our intuition about the double sum. We notice that the inner sum looks like a Riemann sum, which approximates a single integral. In fact, for fixed index i, $f(x_i, y)$ is a continuous function of y so we know that

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(x_i, y_j)\Delta y_j = \int_a^b f(x_i, y)dy$$

Because this limit exists, we again use the definition of the limit at infinity to conclude that, given any small number $\epsilon' > 0$, then for n large enough,

$$\left| \sum_{j=1}^{n} f(x_i, y_j)\Delta y_j - \int_a^b f(x_i, y)dy \right| < \epsilon'.$$

So substituting this result into the expression (1) we have, for any $\epsilon' > 0$

$$\left| \sum_{i=1}^{m} \left(\int_a^b f(x_i, y)dy \right) \Delta x_i - \iint_R f(x, y)dA \right| + \sum_{i=1}^{m} \epsilon' \Delta x_i < \epsilon.$$

Note that

$$\sum_{i=1}^{m} \epsilon' \Delta x_i = (d - c)\epsilon'$$

since the sum over Δx_i is just a partition of the interval $[c, d]$. By choosing ϵ' small enough, we can guarantee that

$$\left| \sum_{i=1}^{m} \left(\int_a^b f(x_i, y)dy \right) \Delta x_i - \iint_R f(x, y)dA \right| < \epsilon$$

for any $\epsilon > 0$.

But this means that by definition,

$$\lim_{m\to\infty} \sum_{i=1}^{m} \left(\int_a^b f(x_i, y)dy \right) \Delta x_i = \iint_R f(x, y)dA.$$

The left-hand side of this equation is a Riemann sum:

$$\lim_{m\to\infty} \sum_{i=1}^{m} \left(\int_a^b f(x_i, y)dy \right) \Delta x_i = \int_c^d \left(\int_a^b f(x_i, y)dy \right) dx$$

Comparing the two equalities, we conclude

$$\iint_R f(x, y)dA = \int_c^d \left(\int_a^b f(x_i, y)dy \right) dx.$$