MULTIVARIABLE AND VECTOR ANALYSIS

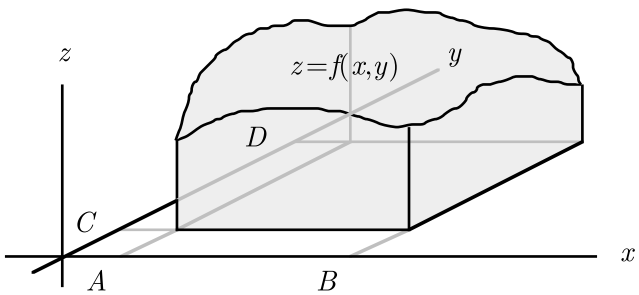
W W L CHEN

Chapter 5

DOUBLE AND TRIPLE INTEGRALS

5.1. Introduction

Consider a real valued function $f(x, y)$, defined over a rectangle $R = [A, B] \times [C, D]$. Suppose, for simplicity, that $f(x, y) \geq 0$ for every $(x, y) \in R$. We would like to evaluate the volume of the region in $\mathbb{R}^3$ above R on the xy-plane (between the planes $x = A$ and $x = B$, and between the planes $y = C$ and $y = D$) and under the surface $z = f(x, y)$.

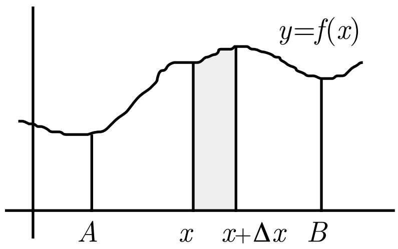


Let this volume be represented by

$$\iint_R f(x, y) \, \mathrm{d}x\mathrm{d}y.$$

The purpose of this chapter is to investigate the properties of this "integral".

We shall first of all take a very cavalier approach to the problem. Consider the simpler case of a function $f(x)$ defined over an interval $[A, B]$. Suppose, for simplicity, that $f(x) \geq 0$ for every $x \in [A, B]$.



Let us split the interval $[A, B]$ into a large number of very short intervals. Consider now one such interval $[x, x + \Delta x]$, where Δx is very small. Then the region in $\mathbb{R}^2$ above the interval $[x, x + \Delta x]$ on the x-axis and under the curve $y = f(x)$ is roughly a rectangle with base Δx and height $f(x)$, and so has area roughly equal to $f(x)\Delta x$. Hence the area of the region in $\mathbb{R}^2$ above the interval $[A, B]$ on the x-axis and under the curve $y = f(x)$ is roughly

$$\sum_{\Delta x} f(x)\Delta x,$$

where the summation is over all these very short intervals making up the interval $[A, B]$. As $\Delta x \to 0$, we have, with any luck,

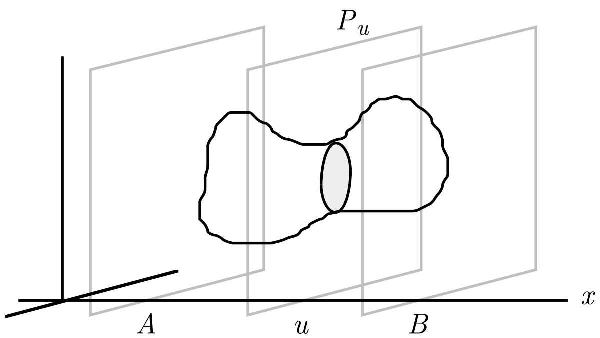
$$\sum_{\Delta x} f(x)\Delta x \to \int_A^B f(x)\,\mathrm{d}x.$$

We next extend this approach to the problem of finding the volume of an object in 3-space. Consistent with our lack of rigour so far, the following seems plausible.

CAVALIERI'S PRINCIPLE. *Suppose that S is a solid in 3-space, and that for $u \in [\alpha, \beta]$, P_u is a family of parallel planes such that the solid S lies between the planes P_α and P_β. Suppose further that for every $u \in [\alpha, \beta]$, the area of the intersection of S with the plane P_u is given by $a(u)$. Then the volume of S is given by*

$$\int_\alpha^\beta a(u)\,\mathrm{d}u.$$

Let us now apply Cavalieri's principle to our original problem. For every $u \in [A, B]$, let P_u denote the plane $x = u$.



Clearly the region in question lies between the planes P_A and P_B. On the other hand, if $a(u)$ denotes the area of the intersection between the region in question and the plane $x = u$, then

$$a(u) = \int_C^D f(u,y)\,\mathrm{d}y.$$

By Cavalieri's principle, the volume of the region in question is now given by

$$\int_A^B a(u)\,\mathrm{d}u = \int_A^B \left(\int_C^D f(u,y)\,\mathrm{d}y\right)\mathrm{d}u = \int_A^B \left(\int_C^D f(x,y)\,\mathrm{d}y\right)\mathrm{d}x.$$

Similarly, for every $u \in [C,D]$, let P_u denote the plane $y = u$. Clearly the region in question lies between the planes P_C and P_D. On the other hand, if $a(u)$ denotes the area of the intersection between the region in question and the plane $y = u$, then

$$a(u) = \int_A^B f(x,u)\,\mathrm{d}x.$$

By Cavalieri's principle, the volume of the region in question is now given by

$$\int_C^D a(u)\,\mathrm{d}u = \int_C^D \left(\int_A^B f(x,u)\,\mathrm{d}x\right)\mathrm{d}u = \int_C^D \left(\int_A^B f(x,y)\,\mathrm{d}x\right)\mathrm{d}y.$$

We therefore conclude that, with any luck,

$$\iint_R f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_A^B \left(\int_C^D f(x,y)\,\mathrm{d}y\right)\mathrm{d}x = \int_C^D \left(\int_A^B f(x,y)\,\mathrm{d}x\right)\mathrm{d}y. \tag{1}$$

Unfortunately, the identity (1) does not hold all the time.

Example 5.1.1. Consider the function $f : [0,1] \times [0,1] \to \mathbb{R}$, given by

$$f(x,y) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 2y & \text{if } x \text{ is irrational,} \end{cases}$$

Then

$$\int_0^1 f(x,y)\,\mathrm{d}y = \begin{cases} \displaystyle\int_0^1 \mathrm{d}y = 1 & \text{if } x \text{ is rational,} \\ \displaystyle\int_0^1 2y\,\mathrm{d}y = 1 & \text{if } x \text{ is irrational,} \end{cases}$$

so that

$$\int_0^1 \left(\int_0^1 f(x,y)\,\mathrm{d}y\right)\mathrm{d}x = 1.$$

On the other hand, the integral

$$\int_0^1 f(x,y)\,\mathrm{d}x$$

does not exist except when $y = 1/2$, so

$$\int_0^1 \left(\int_0^1 f(x,y)\,\mathrm{d}x\right)\mathrm{d}y$$

does not exist.

5.2. Double Integrals over Rectangles

Suppose that the function $f : R \to \mathbb{R}^2$ is bounded in R, where $R = [A, B] \times [C, D]$ is a rectangle. Suppose further that

$$\Delta : \ A = x_0 < x_1 < x_2 < \ldots < x_n = B, \ C = y_0 < y_1 < y_2 < \ldots < y_m = D$$

is a dissection of the rectangle $R = [A, B] \times [C, D]$.

Definition. The sum

$$s(\Delta) = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i - x_{i-1})(y_j - y_{j-1}) \min_{(x,y)\in[x_{i-1},x_i]\times[y_{j-1},y_j]} f(x, y)$$

is called the lower Riemann sum of $f(x, y)$ corresponding to the dissection Δ.

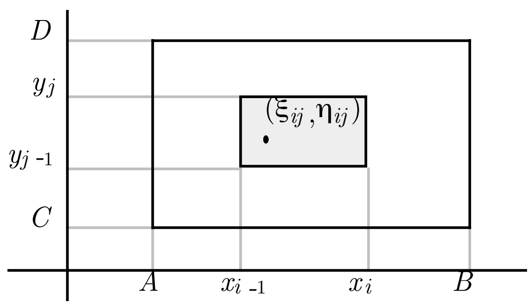
Definition. The sum

$$S(\Delta) = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i - x_{i-1})(y_j - y_{j-1}) \max_{(x,y)\in[x_{i-1},x_i]\times[y_{j-1},y_j]} f(x, y)$$

is called the upper Riemann sum of $f(x, y)$ corresponding to the dissection Δ.

Remark. Strictly speaking, the above definitions are invalid, since the minimum or maximum may not exist. The correct way is to replace the minimum and maximum with infimum and supremum respectively. However, since we have not discussed infimum and supremum, we shall be somewhat economical with the truth and simply use minimum and maximum.

Definition. Suppose that for every $i = 1, \ldots, n$ and $j = 1, \ldots, m$, the point (ξ_{ij}, η_{ij}) lies in the rectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$.



Then the sum

$$\sum_{i=1}^{n} \sum_{j=1}^{m} (x_i - x_{i-1})(y_j - y_{j-1}) f(\xi_{ij}, \eta_{ij})$$

is called a Riemann sum of $f(x, y)$ corresponding to the dissection Δ.

Remarks. (1) It is clear that

$$\min_{(x,y)\in[x_{i-1},x_i]\times[y_{j-1},y_j]} f(x, y) \le f(\xi_{ij}, \eta_{ij}) \le \max_{(x,y)\in[x_{i-1},x_i]\times[y_{j-1},y_j]} f(x, y).$$

It follows that every Riemann sum is bounded below by the corresponding lower Riemann sum and bounded above by the corresponding upper Riemann sum; in other words,

$$s(\Delta) \le \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i - x_{i-1})(y_j - y_{j-1}) f(\xi_{ij}, \eta_{ij}) \le S(\Delta).$$

(2) It can be shown that for any two dissections Δ' and Δ'' of the rectangle $R = [A, B] \times [C, D]$, we have $s(\Delta') \le S(\Delta'')$; in other words, a lower Riemann sum can never exceed an upper Riemann sum.

Definition. We say that

$$\iint_R f(x, y)\,\mathrm{d}x\mathrm{d}y = L$$

if, given any $\epsilon > 0$, there exists a dissection Δ of $R = [A, B] \times [C, D]$ such that

$$L - \epsilon < s(\Delta) \le S(\Delta) < L + \epsilon.$$

In this case, we say that the function $f(x, y)$ is Riemann integrable over the rectangle $R = [A, B] \times [C, D]$ with integral L.

Remark. In other words, if the lower Riemann sums and upper Riemann sums can get arbitrarily close, then their common value is the integral of the function.

The following result follows easily from our definition. The proof is left as an exercise.

THEOREM 5A. *Suppose that the functions $f : R \to \mathbb{R}$ and $g : R \to \mathbb{R}$ are Riemann integrable over the rectangle $R = [A, B] \times [C, D]$.*
(a) Then the function $f + g$ is Riemann integrable over R, and

$$\iint_R (f(x, y) + g(x, y))\,\mathrm{d}x\mathrm{d}y = \iint_R f(x, y)\,\mathrm{d}x\mathrm{d}y + \iint_R g(x, y)\,\mathrm{d}x\mathrm{d}y;$$

(b) On the other hand, for any $c \in \mathbb{R}$, the function cf is Riemann integrable over R, and

$$\iint_R cf(x, y)\,\mathrm{d}x\mathrm{d}y = c \iint_R f(x, y)\,\mathrm{d}x\mathrm{d}y.$$

(c) Suppose further that $f(x, y) \ge g(x, y)$ for every $(x, y) \in R$. Then

$$\iint_R f(x, y)\,\mathrm{d}x\mathrm{d}y \ge \iint_R g(x, y)\,\mathrm{d}x\mathrm{d}y.$$

We also state without proof the following result.

THEOREM 5B. *Suppose that $Q = R_1 \cup \ldots \cup R_p$ is a rectangle, where, for $k = 1, \ldots, p$, the rectangles $R_k = [A_k, B_k] \times [C_k, D_k]$ are pairwise disjoint, apart possibly from boundary points. Suppose further that the function $f : Q \to \mathbb{R}$ is bounded in Q. Then f is Riemann integrable over Q if and only if it is Riemann integrable over R_k for every $k = 1, \ldots, p$. In this case, we also have*

$$\iint_Q f(x, y)\,\mathrm{d}x\mathrm{d}y = \sum_{k=1}^{p} \iint_{R_k} f(x, y)\,\mathrm{d}x\mathrm{d}y.$$

5.3. Conditions for Integrability

The following important result will be discussed in Section 5.5.

THEOREM 5C. (FUBINI'S THEOREM) *Suppose that the function $f : R \to \mathbb{R}$ is continuous in the rectangle $R = [A, B] \times [C, D]$. Then f is Riemann integrable over R. Furthermore, the identity* (1) *holds.*

EXAMPLE 5.3.1. Consider the integral

$$\iint_R (x^2 + y^2)\,\mathrm{d}x\mathrm{d}y,$$

where $R = [0, 2] \times [0, 1]$ is a rectangle. Since the function $f(x, y) = x^2 + y^2$ is continuous in R, the integral exists by Fubini's theorem. Furthermore,

$$\int_0^2 \left(\int_0^1 (x^2 + y^2)\,\mathrm{d}y \right) \mathrm{d}x = \int_0^2 \left(x^2 + \frac{1}{3} \right) \mathrm{d}x = \frac{10}{3}.$$

On the other hand,

$$\int_0^1 \left(\int_0^2 (x^2 + y^2)\,\mathrm{d}x \right) \mathrm{d}y = \int_0^1 \left(\frac{8}{3} + 2y^2 \right) \mathrm{d}y = \frac{10}{3}.$$

By Fubini's theorem,

$$\iint_R (x^2 + y^2)\,\mathrm{d}x\mathrm{d}y = \frac{10}{3}.$$

EXAMPLE 5.3.2. Consider the integral

$$\iint_R \sin x \cos y\,\mathrm{d}x\mathrm{d}y,$$

where $R = [0, \pi] \times [-\pi/2, \pi/2]$ is a rectangle. Since the function $f(x, y) = \sin x \cos y$ is continuous in R, the integral exists by Fubini's theorem. Furthermore,

$$\int_0^\pi \left(\int_{-\pi/2}^{\pi/2} \sin x \cos y\,\mathrm{d}y \right) \mathrm{d}x = \int_0^\pi 2\sin x\,\mathrm{d}x = 4.$$

On the other hand,

$$\int_{-\pi/2}^{\pi/2} \left(\int_0^\pi \sin x \cos y\,\mathrm{d}x \right) \mathrm{d}y = \int_{-\pi/2}^{\pi/2} 2\cos y\,\mathrm{d}y = 4.$$

By Fubini's theorem,

$$\iint_R \sin x \cos y\,\mathrm{d}x\mathrm{d}y = 4.$$

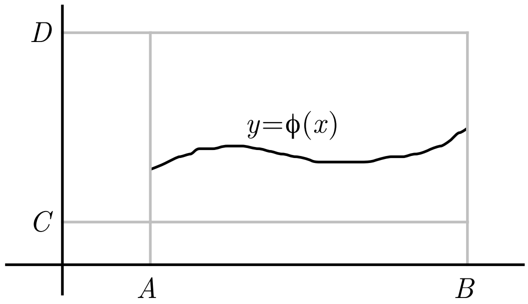
5.4. Double Integrals over Special Regions

The purpose of this section is to study Riemann integration over regions R which are not necessarily rectangles of the form $[A, B] \times [C, D]$. Our argument hinges on the following generalization of Fubini's theorem.

Definition. Consider the rectangle $R = [A, B] \times [C, D]$. A set of the form

$$\{(x, \phi(x)) : x \in [A, B]\} \subseteq R$$

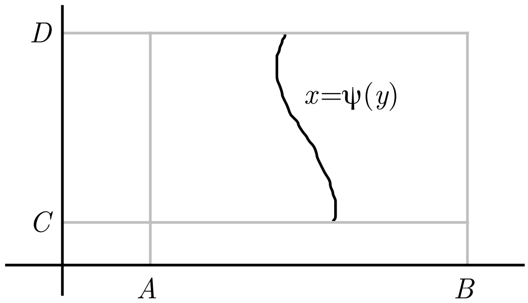
is said to be a curve of type 1 in R if the function $\phi : [A, B] \to \mathbb{R}$ is continuous in the interval $[A, B]$.



A set of the form

$$\{(\psi(y), y) : y \in [C, D]\} \subseteq R$$

is said to be a curve of type 2 in R if the function $\psi : [C, D] \to \mathbb{R}$ is continuous in the interval $[C, D]$.



THEOREM 5D. *Suppose that the function $f : R \to \mathbb{R}$ is bounded in the rectangle $R = [A, B] \times [C, D]$. Suppose further that f is continuous in R, except possibly at points contained in a finite number of curves of type 1 or 2 in R. Then f is Riemann integrable over R. Furthermore, if the integral*

$$\int_C^D f(x, y)\,\mathrm{d}y$$

exists for every $x \in [A, B]$, then the integral

$$\int_A^B \left(\int_C^D f(x, y)\,\mathrm{d}y \right) \mathrm{d}x$$

exists, and

$$\iint_R f(x, y)\,\mathrm{d}x\mathrm{d}y = \int_A^B \left(\int_C^D f(x, y)\,\mathrm{d}y \right) \mathrm{d}x.$$

Similarly, if the integral

$$\int_A^B f(x, y)\,\mathrm{d}x$$

exists for every $y \in [C,D]$*, then the integral*

$$\int_C^D \left(\int_A^B f(x,y)\,\mathrm{d}x \right) \mathrm{d}y$$

exists, and

$$\iint_R f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_C^D \left(\int_A^B f(x,y)\,\mathrm{d}x \right) \mathrm{d}y.$$

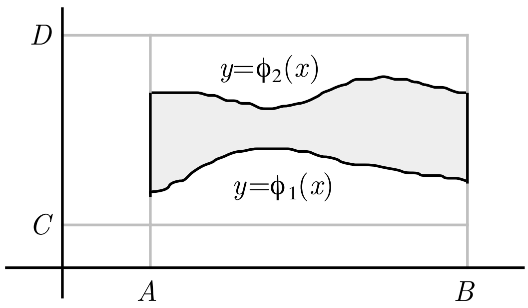
Thus, the identity (1) *holds if all the integrals exist.*

In view of Theorem 5D, we can now study integrals over some special regions.

DEFINITION. Suppose that $\phi_1 : [A,B] \to \mathbb{R}$ and $\phi_2 : [A,B] \to \mathbb{R}$ are continuous in the interval $[A,B]$. Suppose further that $\phi_1(x) \le \phi_2(x)$ for every $x \in [A,B]$. Then we say that

$$S = \{(x,y) \in \mathbb{R}^2 : x \in [A,B] \text{ and } \phi_1(x) \le y \le \phi_2(x)\} \tag{2}$$

is an elementary region of type 1.



Suppose now that S is an elementary region of type 1, of the form (2). Since $\phi_1 : [A,B] \to \mathbb{R}$ and $\phi_2 : [A,B] \to \mathbb{R}$ are continuous in $[A,B]$, there exist $C, D \in \mathbb{R}$ such that $C \le \phi_1(x) \le \phi_2(x) \le D$ for every $x \in [A,B]$. It follows that $S \subseteq R$, where $R = [A,B] \times [C,D]$. Suppose that the function $f : S \to \mathbb{R}$ is continuous in S. Then it is bounded in S. Hence the function $f^* : R \to \mathbb{R}$, defined by

$$f^*(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in S, \\ 0 & \text{if } (x,y) \notin S, \end{cases}$$

is bounded in R. Furthermore, it is continuous in R, except possibly at points contained in two curves of type 1 in R. It follows from Theorem 5D that f^* is Riemann integrable over R. We can now define

$$\iint_S f(x,y)\,\mathrm{d}x\mathrm{d}y = \iint_R f^*(x,y)\,\mathrm{d}x\mathrm{d}y.$$

On the other hand, for any $x \in [A,B]$, the function $f^*(x,y) = f(x,y)$ is a continuous function of y in the interval $[\phi_1(x), \phi_2(x)]$. Also, $f^*(x,y) = 0$ for every $y \in [C, \phi_1(x)) \cup (\phi_2(x), D]$. Hence

$$\int_C^D f^*(x,y)\,\mathrm{d}y = \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,\mathrm{d}y,$$

and so

$$\int_A^B \left(\int_C^D f^*(x,y)\,\mathrm{d}y \right) \mathrm{d}x = \int_A^B \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$$

We have proved the following result.

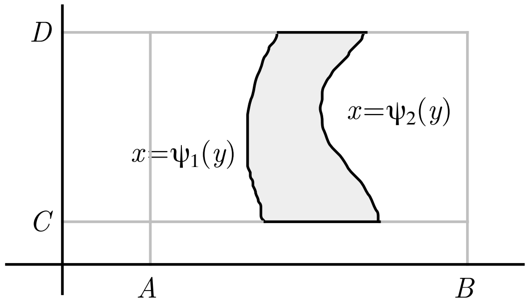
THEOREM 5E. *Suppose that S is an elementary region of type 1, of the form (2). Suppose further that the function $f : S \to \mathbb{R}$ is continuous in S. Then*

$$\iint_S f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_A^B \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$$

DEFINITION. Suppose that $\psi_1 : [C,D] \to \mathbb{R}$ and $\psi_2 : [C,D] \to \mathbb{R}$ are continuous in the interval $[C,D]$. Suppose further that $\psi_1(y) \le \psi_2(y)$ for every $y \in [C,D]$. Then we say that

(3) $$S = \{(x,y) \in \mathbb{R}^2 : y \in [C,D] \text{ and } \psi_1(y) \le x \le \psi_2(y)\}$$

is an elementary region of type 2.



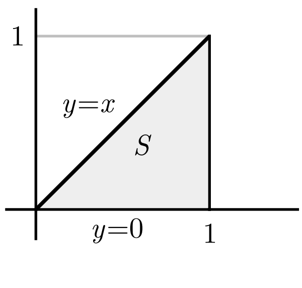
Analogous to Theorem 5E, we have the following result.

THEOREM 5F. *Suppose that S is an elementary region of type 2, of the form (3). Suppose further that the function $f : S \to \mathbb{R}$ is continuous in S. Then*

$$\iint_S f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_C^D \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y.$$

EXAMPLE 5.4.1. The function $f(x,y) = x^2 + y^2$ is continuous in the triangle S with vertices $(0,0)$, $(1,0)$ and $(1,1)$. It is easy to see that S is an elementary region of type 1, of the form

$$S = \{(x,y) \in \mathbb{R}^2 : x \in [0,1] \text{ and } 0 \le y \le x\}.$$

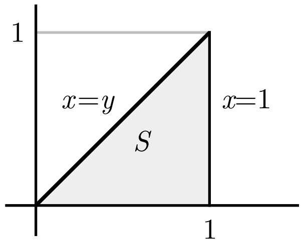


By Theorem 5E,

$$\iint_S (x^2+y^2)\,\mathrm{d}x\mathrm{d}y = \int_0^1 \left(\int_0^x (x^2+y^2)\,\mathrm{d}y \right) \mathrm{d}x = \int_0^1 \frac{4}{3}x^3\,\mathrm{d}x = \frac{1}{3}.$$

On the other hand, it is also easy to see that S is an elementary region of type 2, of the form

$$S = \{(x,y) \in \mathbb{R}^2 : y \in [0,1] \text{ and } y \le x \le 1\}.$$

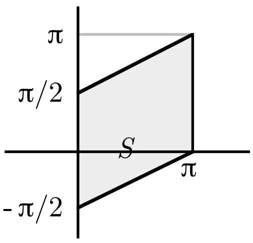


By Theorem 5F,

$$\iint_S (x^2+y^2)\,\mathrm{d}x\mathrm{d}y = \int_0^1 \left(\int_y^1 (x^2+y^2)\,\mathrm{d}x \right) \mathrm{d}y = \int_0^1 \left(\frac{1}{3} + y^2 - \frac{4}{3}y^3 \right) \mathrm{d}y = \frac{1}{3}.$$

EXAMPLE 5.4.2. The function $f(x,y) = \sin x \cos y$ is continuous in the parallelogram S with vertices $(\pi,0)$, (π,π) and $(0,\pm\pi/2)$. It is easy to see that S is an elementary region of type 1, of the form

$$S = \left\{ (x,y) \in \mathbb{R}^2 : x \in [0,\pi] \text{ and } \frac{x}{2} - \frac{\pi}{2} \le y \le \frac{x}{2} + \frac{\pi}{2} \right\}.$$



By Theorem 5E,

$$\begin{aligned}
\iint_S \sin x \cos y\,\mathrm{d}x\mathrm{d}y &= \int_0^\pi \left(\int_{\frac{x}{2}-\frac{\pi}{2}}^{\frac{x}{2}+\frac{\pi}{2}} \sin x \cos y\,\mathrm{d}y \right) \mathrm{d}x = \int_0^\pi \sin x \left(\sin\left(\frac{x}{2}+\frac{\pi}{2}\right) - \sin\left(\frac{x}{2}-\frac{\pi}{2}\right) \right) \mathrm{d}x \\
&= \int_0^\pi \sin x \left(\sin\frac{x}{2}\cos\frac{\pi}{2} + \cos\frac{x}{2}\sin\frac{\pi}{2} - \sin\frac{x}{2}\cos\frac{\pi}{2} + \cos\frac{x}{2}\sin\frac{\pi}{2} \right) \mathrm{d}x \\
&= 2\int_0^\pi \sin x \cos\frac{x}{2}\,\mathrm{d}x = 4\int_0^\pi \sin\frac{x}{2}\cos^2\frac{x}{2}\,\mathrm{d}x = \frac{8}{3}.
\end{aligned}$$

On the other hand, it is also easy to see that S is an elementary region of type 2, of the form

$$S = \left\{ (x,y) \in \mathbb{R}^2 : y \in \left[-\frac{\pi}{2}, \pi\right] \text{ and } \psi_1(y) \le x \le \psi_2(y) \right\},$$

where

$$\psi_1(y) = \begin{cases} 0 & \text{if } y \in [-\pi/2, \pi/2], \\ 2y - \pi & \text{if } y \in [\pi/2, \pi], \end{cases} \qquad \text{and} \qquad \psi_2(y) = \begin{cases} 2y + \pi & \text{if } y \in [-\pi/2, 0], \\ \pi & \text{if } y \in [0, \pi]. \end{cases}$$

By Theorem 5F,

$$\begin{aligned}
&\iint_S \sin x \cos y \,\mathrm{d}x\mathrm{d}y \\
&= \int_{-\pi/2}^{0} \left(\int_0^{2y+\pi} \sin x \cos y \,\mathrm{d}x \right) \mathrm{d}y + \int_0^{\pi/2} \left(\int_0^{\pi} \sin x \cos y \,\mathrm{d}x \right) \mathrm{d}y + \int_{\pi/2}^{\pi} \left(\int_{2y-\pi}^{\pi} \sin x \cos y \,\mathrm{d}x \right) \mathrm{d}y \\
&= \int_{-\pi/2}^{0} (\cos y - \cos(2y+\pi)\cos y)\,\mathrm{d}y + \int_0^{\pi/2} 2\cos y \,\mathrm{d}y + \int_{\pi/2}^{\pi} (\cos y + \cos(2y-\pi)\cos y)\,\mathrm{d}y \\
&= \int_{-\pi/2}^{0} \cos y \,\mathrm{d}y + 2\int_0^{\pi/2} \cos y \,\mathrm{d}y + \int_{\pi/2}^{\pi} \cos y \,\mathrm{d}y - \int_{-\pi/2}^{0} \cos(2y+\pi)\cos y \,\mathrm{d}y + \int_{\pi/2}^{\pi} \cos(2y-\pi)\cos y \,\mathrm{d}y \\
&= 2 - \int_{-\pi/2}^{0} (\cos 2y \cos \pi - \sin 2y \sin \pi)\cos y \,\mathrm{d}y + \int_{\pi/2}^{\pi} (\cos 2y \cos \pi + \sin 2y \sin \pi)\cos y \,\mathrm{d}y \\
&= 2 + \int_{-\pi/2}^{0} \cos 2y \cos y \,\mathrm{d}y - \int_{\pi/2}^{\pi} \cos 2y \cos y \,\mathrm{d}y \\
&= 2 + \int_{-\pi/2}^{0} (1 - 2\sin^2 y)\cos y \,\mathrm{d}y - \int_{\pi/2}^{\pi} (1 - 2\sin^2 y)\cos y \,\mathrm{d}y \\
&= 2 + \int_{-\pi/2}^{0} \cos y \,\mathrm{d}y - \int_{\pi/2}^{\pi} \cos y \,\mathrm{d}y - 2\int_{-\pi/2}^{0} \sin^2 y \cos y \,\mathrm{d}y + 2\int_{\pi/2}^{\pi} \sin^2 y \cos y \,\mathrm{d}y = \frac{8}{3}.
\end{aligned}$$

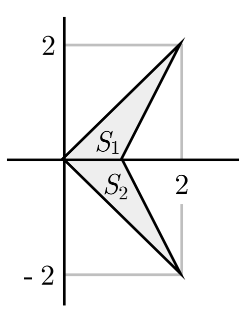
Note that the calculation is much simpler if we think of S as an elementary region of type 1.

We can extend our study further. Suppose that $T = S_1 \cup \ldots \cup S_p$ is a finite region in $\mathbb{R}^2$, where, for every $k = 1, \ldots, p$, S_k is an elementary region of type 1 or 2. Suppose further that $S_1, \ldots, S_p$ are pairwise disjoint, apart possibly from boundary points. Then we can define

$$\iint_T f(x,y)\,\mathrm{d}x\mathrm{d}y = \sum_{k=1}^{p} \iint_{S_k} f(x,y)\,\mathrm{d}x\mathrm{d}y,$$

provided that every integral on the right hand side exists.

Example 5.4.3. Consider the finite region T bounded by the four lines $y = x$, $y = -x$, $y = 2x - 2$ and $y = 2 - 2x$ and with vertices $(0,0)$, $(2,-2)$, $(1,0)$ and $(2,2)$.



Note that $T = S_1 \cup S_2$, where S_1 is the triangle with vertices $(0,0)$, $(1,0)$ and $(2,2)$ and S_2 is the triangle with vertices $(0,0)$, $(1,0)$ and $(2,-2)$. It is easy to see that S_1 and S_2 are disjoint, apart from boundary points. Clearly the function $f(x,y) = x + y^2$ is continuous in T, and so continuous in S_1 and S_2. Hence

$$\iint_T (x + y^2)\,\mathrm{d}x\mathrm{d}y = \iint_{S_1} (x + y^2)\,\mathrm{d}x\mathrm{d}y + \iint_{S_2} (x + y^2)\,\mathrm{d}x\mathrm{d}y.$$

To study the first integral on the right hand side, it is easier to interpret S_1 as an elementary region of type 2, of the form

$$S_1 = \left\{(x,y) \in \mathbb{R}^2 : y \in [0,2] \text{ and } y \le x \le 1 + \frac{y}{2}\right\}.$$

Then

$$\begin{aligned}\iint_{S_1} (x+y^2)\,\mathrm{d}x\mathrm{d}y &= \int_0^2 \left(\int_y^{1+\frac{y}{2}} (x+y^2)\,\mathrm{d}x\right)\mathrm{d}y = \int_0^2 \left(\frac{1}{2}\left(1+\frac{y}{2}\right)^2 + \left(1+\frac{y}{2}\right)y^2 - \frac{1}{2}y^2 - y^3\right)\mathrm{d}y \\ &= \int_0^2 \left(\frac{1}{2} + \frac{1}{2}y + \frac{5}{8}y^2 - \frac{1}{2}y^3\right)\mathrm{d}y = \frac{5}{3}.\end{aligned}$$

To study the second integral on the right hand side, it is again easier to interpret S_2 as an elementary region of type 2, of the form

$$S_2 = \left\{(x,y) \in \mathbb{R}^2 : y \in [-2,0] \text{ and } -y \le x \le 1 - \frac{y}{2}\right\}.$$

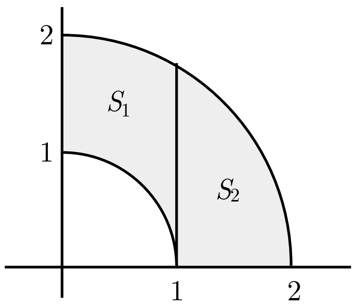
Then

$$\begin{aligned}\iint_{S_2} (x+y^2)\,\mathrm{d}x\mathrm{d}y &= \int_{-2}^0 \left(\int_{-y}^{1-\frac{y}{2}} (x+y^2)\,\mathrm{d}x\right)\mathrm{d}y = \int_{-2}^0 \left(\frac{1}{2}\left(1-\frac{y}{2}\right)^2 + \left(1-\frac{y}{2}\right)y^2 - \frac{1}{2}y^2 + y^3\right)\mathrm{d}y \\ &= \int_{-2}^0 \left(\frac{1}{2} - \frac{1}{2}y + \frac{5}{8}y^2 + \frac{1}{2}y^3\right)\mathrm{d}y = \frac{5}{3}.\end{aligned}$$

Hence

$$\iint_T (x+y^2)\,\mathrm{d}x\mathrm{d}y = \frac{10}{3}.$$

Example 5.4.4. Consider the region T in the first quadrant between the two circles of radii 1 and 2 and centred at $(0,0)$.



Note that $T = S_1 \cup S_2$, where

$$S_1 = \{(x,y) \in \mathbb{R}^2 : x \in [0,1] \text{ and } (1-x^2)^{1/2} \le y \le (4-x^2)^{1/2}\}$$

and

$$S_2 = \{(x,y) \in \mathbb{R}^2 : x \in [1,2] \text{ and } 0 \le y \le (4-x^2)^{1/2}\}$$

are elementary regions of type 1 and disjoint, apart from boundary points. The function $f(x,y) = x^2+y^2$ is clearly continuous in T, and so continuous in S_1 and S_2. Hence

$$\iint_T (x^2+y^2)\,\mathrm{d}x\mathrm{d}y = \iint_{S_1} (x^2+y^2)\,\mathrm{d}x\mathrm{d}y + \iint_{S_2} (x^2+y^2)\,\mathrm{d}x\mathrm{d}y.$$

We have

$$\begin{aligned}\iint_{S_1} (x^2+y^2)\,\mathrm{d}x\mathrm{d}y &= \int_0^1 \left(\int_{(1-x^2)^{1/2}}^{(4-x^2)^{1/2}} (x^2+y^2)\,\mathrm{d}y \right) \mathrm{d}x \\ &= \int_0^1 \left(x^2(4-x^2)^{1/2} + \frac{1}{3}(4-x^2)^{3/2} - x^2(1-x^2)^{1/2} - \frac{1}{3}(1-x^2)^{3/2} \right) \mathrm{d}x \\ &= \frac{4}{3}\int_0^1 (4-x^2)^{1/2}\,\mathrm{d}x + \frac{2}{3}\int_0^1 x^2(4-x^2)^{1/2}\,\mathrm{d}x - \frac{1}{3}\int_0^1 (1-x^2)^{1/2}\,\mathrm{d}x - \frac{2}{3}\int_0^1 x^2(1-x^2)^{1/2}\,\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\iint_{S_2} (x^2+y^2)\,\mathrm{d}x\mathrm{d}y &= \int_1^2 \left(\int_0^{(4-x^2)^{1/2}} (x^2+y^2)\,\mathrm{d}y \right) \mathrm{d}x \\ &= \int_1^2 \left(x^2(4-x^2)^{1/2} + \frac{1}{3}(4-x^2)^{3/2} \right) \mathrm{d}x = \frac{4}{3}\int_1^2 (4-x^2)^{1/2}\,\mathrm{d}x + \frac{2}{3}\int_1^2 x^2(4-x^2)^{1/2}\,\mathrm{d}x.\end{aligned}$$

Hence

$$\begin{aligned}&\iint_T (x^2+y^2)\,\mathrm{d}x\mathrm{d}y \\ &\quad = \frac{4}{3}\int_0^2 (4-x^2)^{1/2}\,\mathrm{d}x + \frac{2}{3}\int_0^2 x^2(4-x^2)^{1/2}\,\mathrm{d}x - \frac{1}{3}\int_0^1 (1-x^2)^{1/2}\,\mathrm{d}x - \frac{2}{3}\int_0^1 x^2(1-x^2)^{1/2}\,\mathrm{d}x \\ &\quad = \frac{16}{3}\int_0^1 (1-z^2)^{1/2}\,\mathrm{d}z + \frac{32}{3}\int_0^1 z^2(1-z^2)^{1/2}\,\mathrm{d}z - \frac{1}{3}\int_0^1 (1-x^2)^{1/2}\,\mathrm{d}x - \frac{2}{3}\int_0^1 x^2(1-x^2)^{1/2}\,\mathrm{d}x \\ &\quad = 5\int_0^1 (1-x^2)^{1/2}\,\mathrm{d}x + 10\int_0^1 x^2(1-x^2)^{1/2}\,\mathrm{d}x = \frac{15\pi}{8}.\end{aligned}$$

Sometimes, we may be faced with repeated integrals that are extremely difficult to handle. Occasionally, a change in the order of integration may be helpful. We illustrate this point by the following two examples.

EXAMPLE 5.4.5. Consider the repeated integral

$$\int_0^{\sqrt{2}} \left(\int_{y/2}^{1/\sqrt{2}} \cos(\pi x^2)\,\mathrm{d}x \right) \mathrm{d}y. \tag{4}$$

Here the inner integral

$$\int_{y/2}^{1/\sqrt{2}} \cos(\pi x^2)\,\mathrm{d}x$$

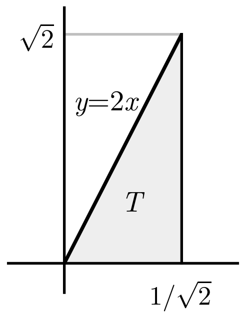
is rather hard to evaluate. However, we may treat the integral (4) as a double integral of the form

$$\iint_S \cos(\pi x^2)\,\mathrm{d}x\mathrm{d}y.$$

To make any progress, we must first of all find out what the region S is. Clearly

$$S = \left\{ (x,y) \in \mathbb{R}^2 : y \in [0,\sqrt{2}] \text{ and } \frac{y}{2} \le x \le \frac{1}{\sqrt{2}} \right\}$$

is a triangle as shown below.



Interchanging the order of integration, we may interpret the region S as an elementary region of type 1, of the form

$$S = \left\{ (x,y) \in \mathbb{R}^2 : x \in \left[0, \frac{1}{\sqrt{2}}\right] \text{ and } 0 \le y \le 2x \right\},$$

so that

$$\iint_S \cos(\pi x^2)\,\mathrm{d}x\mathrm{d}y = \int_0^{1/\sqrt{2}} \left(\int_0^{2x} \cos(\pi x^2)\,\mathrm{d}y \right) \mathrm{d}x = \int_0^{1/\sqrt{2}} 2x\cos(\pi x^2)\,\mathrm{d}x = \frac{1}{\pi}.$$

EXAMPLE 5.4.6. Suppose that we are asked to evaluate

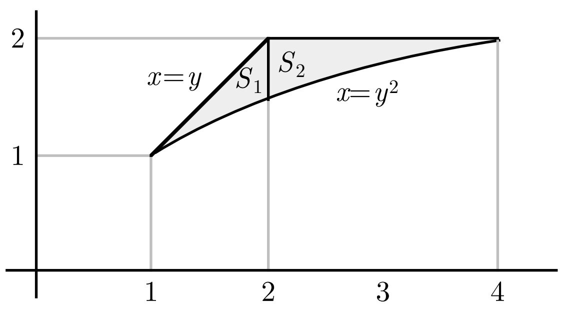
$$\int_1^2 \left(\int_{\sqrt{x}}^x \sin\frac{\pi x}{2y}\,\mathrm{d}y \right) \mathrm{d}x + \int_2^4 \left(\int_{\sqrt{x}}^2 \sin\frac{\pi x}{2y}\,\mathrm{d}y \right) \mathrm{d}x. \tag{5}$$

Here the inner integrals are rather hard to evaluate. Instead, we write

$$\int_1^2 \left(\int_{\sqrt{x}}^x \sin\frac{\pi x}{2y}\,\mathrm{d}y \right) \mathrm{d}x = \iint_{S_1} \sin\frac{\pi x}{2y}\,\mathrm{d}x\mathrm{d}y \qquad \text{and} \qquad \int_2^4 \left(\int_{\sqrt{x}}^2 \sin\frac{\pi x}{2y}\,\mathrm{d}y \right) \mathrm{d}x = \iint_{S_2} \sin\frac{\pi x}{2y}\,\mathrm{d}x\mathrm{d}y,$$

where

$$S_1 = \{(x,y) \in \mathbb{R}^2 : x \in [1,2] \text{ and } \sqrt{x} \le y \le x\} \quad \text{and} \quad S_2 = \{(x,y) \in \mathbb{R}^2 : x \in [2,4] \text{ and } \sqrt{x} \le y \le 2\}.$$



If we write $S = S_1 \cup S_2$, then

$$S = \{(x,y) \in \mathbb{R}^2 : y \in [1,2] \text{ and } y \le x \le y^2\}$$

is an elementary region of type 2, and the sum (5) is equal to

$$\iint_S \sin\frac{\pi x}{2y}\,\mathrm{d}x\mathrm{d}y = \int_1^2\left(\int_y^{y^2}\sin\frac{\pi x}{2y}\,\mathrm{d}x\right)\mathrm{d}y = -\int_1^2 \frac{2y}{\pi}\cos\frac{\pi y}{2}\,\mathrm{d}y = \frac{4(\pi+2)}{\pi^3},$$

where the last step involves integration by parts.

5.5. Fubini's Theorem

In this section, we briefly indicate how continuity of a function f in a rectangle $R = [A,B]\times[C,D]$ leads to integrability of f in R. Note, however, that our discussion here falls well short of a proof of Fubini's theorem. We shall only discuss the following result.

THEOREM 5G. *Suppose that the function $f : R \to \mathbb{R}$ is continuous in the rectangle $R = [A,B]\times[C,D]$. Then for every $\epsilon > 0$, there exists a dissection*

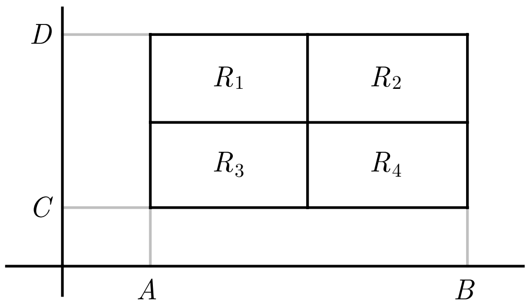
$$\Delta:\ A = x_0 < x_1 < x_2 < \ldots < x_n = B,\ C = y_0 < y_1 < y_2 < \ldots < y_m = D$$

of R such that for every $i = 1,\ldots,n$ and $j = 1,\ldots,m$, we have

$$\max_{(x,y)\in[x_{i-1},x_i]\times[y_{j-1},y_j]} f(x,y) - \min_{(x,y)\in[x_{i-1},x_i]\times[y_{j-1},y_j]} f(x,y) < \epsilon, \tag{6}$$

so that $S(\Delta) - s(\Delta) < (B-A)(D-C)\epsilon$.

† Sketch of Proof. Suppose on the contrary that there exists $\epsilon > 0$ for which no dissection of R will satisfy (6). We now dissect the rectangle R into four similar rectangles as shown.



Then for at least one of the four rectangles R_k $(k = 1,2,3,4)$, we cannot find a dissection of R_k which will achieve an inequality of the type (6). We now dissect this rectangle R_k into four smaller and similar rectangles in the same way. Note that each dissection quarters the area of the rectangle, so this process eventually collapses to a point $(\alpha,\beta)\in R$. Since f is continuous at (α,β) (with a slightly modified argument if (α,β) is on the boundary of R), there exists $\delta > 0$ such that $|f(x,y) - f(\alpha,\beta)| < \epsilon/2$ for every $(x,y)\in R$ satisfying $\|(x,y)-(\alpha,\beta)\| < \delta$. It follows that for every

$$(x_1,y_1),(x_2,y_2)\in\{(x,y)\in R : \|(x,y)-(\alpha,\beta)\| < \delta\},$$

we have

$$|f(x_1,y_1) - f(x_2,y_2)| \le |f(x_1,y_1) - f(\alpha,\beta)| + |f(x_2,y_2) - f(\alpha,\beta)| < \epsilon.$$

However, our dissection will result in a rectangle contained in $\{(x,y)\in R : \|(x,y)-(\alpha,\beta)\| < \delta\}$, giving a contradiction. ○

5.6. Mean Value Theorem

Suppose that $S \subseteq \mathbb{R}^2$ is an elementary region. Suppose further that a function $f : S \to \mathbb{R}$ is continuous in S. Then there exist $(x_1, y_1), (x_2, y_2) \in S$ such that

$$f(x_1, y_1) \leq f(x, y) \leq f(x_2, y_2)$$

for every $(x, y) \in S$; in other words, f has a minimum value and a maximum value in S. It follows that

$$f(x_1, y_1)A(S) = \iint_S f(x_1, y_1)\,\mathrm{d}x\mathrm{d}y \leq \iint_S f(x, y)\,\mathrm{d}x\mathrm{d}y \leq \iint_S f(x_2, y_2)\,\mathrm{d}x\mathrm{d}y = f(x_2, y_2)A(S),$$

where

$$A(S) = \iint_S \mathrm{d}x\mathrm{d}y$$

denotes the area of S, so that

$$f(x_1, y_1) \leq \frac{1}{A(S)} \iint_S f(x, y)\,\mathrm{d}x\mathrm{d}y \leq f(x_2, y_2).$$

Since f is continuous in S, it follows from the Intermediate value theorem that there exists $(x_0, y_0) \in S$ such that

$$f(x_0, y_0) = \frac{1}{A(S)} \iint_S f(x, y)\,\mathrm{d}x\mathrm{d}y.$$

We have sketched a proof of the following result.

THEOREM 5H. (MEAN VALUE THEOREM) *Suppose that $S \subseteq \mathbb{R}^2$ is an elementary region. Suppose further that a function $f : S \to \mathbb{R}$ is continuous in S. Then there exists $(x_0, y_0) \in S$ such that*

$$\iint_S f(x, y)\,\mathrm{d}x\mathrm{d}y = f(x_0, y_0)A(S),$$

where $A(S)$ denotes the area of S.

EXAMPLE 5.6.1. The function $f(x, y) = \cos(x+y)$ has maximum value $f(0, 0) = 1$ and minimum value $f(\pi/4, \pi/4) = 0$ in the rectangle $R = [0, \pi/4] \times [0, \pi/4]$. On the other hand,

$$\begin{aligned}
\iint_R f(x, y)\,\mathrm{d}x\mathrm{d}y &= \int_0^{\pi/4} \left(\int_0^{\pi/4} \cos(x + y)\,\mathrm{d}y \right) \mathrm{d}x = \int_0^{\pi/4} \left(\sin\left(x + \frac{\pi}{4}\right) - \sin x \right) \mathrm{d}x \\
&= \int_0^{\pi/4} \left(\sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4} - \sin x \right) \mathrm{d}x = \int_0^{\pi/4} \left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x - \sin x \right) \mathrm{d}x \\
&= \int_0^{\pi/4} \left(\frac{1}{\sqrt{2}} \cos x - \left(1 - \frac{1}{\sqrt{2}}\right) \sin x \right) \mathrm{d}x = \frac{1}{\sqrt{2}} \left(\sin \frac{\pi}{4} - \sin 0 \right) + \left(1 - \frac{1}{\sqrt{2}}\right) \left(\cos \frac{\pi}{4} - \cos 0 \right) \\
&= \frac{1}{2} + \left(1 - \frac{1}{\sqrt{2}}\right) \left(\frac{1}{\sqrt{2}} - 1 \right) = \sqrt{2} - 1.
\end{aligned}$$

It follows that

$$\frac{1}{A(R)} \iint_R f(x, y)\,\mathrm{d}x\mathrm{d}y = \frac{16}{\pi^2}(\sqrt{2} - 1).$$

Simple calculation gives

$$0 < \frac{16}{\pi^2}(\sqrt{2}-1) < 1,$$

so that there exists $u_0 \in [0,\pi/4]$, and so $(u_0,u_0) \in [0,\pi/4] \times [0,\pi/4]$, such that

$$f(u_0,u_0) = \cos 2u_0 = \frac{16}{\pi^2}(\sqrt{2}-1).$$

EXAMPLE 5.6.2. The function $f(x,y) = x^2+y^2+2$ has maximum value $f(1,1) = 3$ and minimum value $f(0,0) = 2$ in the disc $S = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\}$. Hence

$$2\pi \le \iint_S f(x,y)\,\mathrm{d}x\mathrm{d}y \le 3\pi.$$

5.7. Triple Integrals

In this last section, we extend our discussion so far to the case of triple integrals. Triple integrals over rectangular boxes $[A,B] \times [C,D] \times [P,Q]$ can be studied via Riemann sums if we extend the argument in Section 5.2 to the 3-dimensional case, although it is harder to visualize geometrically the graphs of real valued functions of three real variables.

We have the following 3-dimensional version of Theorem 5C.

THEOREM 5J. (FUBINI'S THEOREM) *Suppose that the function $f : R \to \mathbb{R}$ is continuous in the rectangular box $R = [A,B] \times [C,D] \times [P,Q]$. Then f is Riemann integrable over R. Furthermore,*

$$\begin{aligned}\iiint_R f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_A^B \left(\int_C^D \left(\int_P^Q f(x,y,z)\,\mathrm{d}z\right)\mathrm{d}y\right)\mathrm{d}x = \int_A^B \left(\int_P^Q \left(\int_C^D f(x,y,z)\,\mathrm{d}y\right)\mathrm{d}z\right)\mathrm{d}x \\ &= \int_C^D \left(\int_A^B \left(\int_P^Q f(x,y,z)\,\mathrm{d}z\right)\mathrm{d}x\right)\mathrm{d}y = \int_C^D \left(\int_P^Q \left(\int_A^B f(x,y,z)\,\mathrm{d}x\right)\mathrm{d}z\right)\mathrm{d}y \\ &= \int_P^Q \left(\int_C^D \left(\int_A^B f(x,y,z)\,\mathrm{d}x\right)\mathrm{d}y\right)\mathrm{d}z = \int_P^Q \left(\int_A^B \left(\int_C^D f(x,y,z)\,\mathrm{d}y\right)\mathrm{d}x\right)\mathrm{d}z.\end{aligned}$$

EXAMPLE 5.7.1. The function $f(x,y,z) = \cos x - \sin y - \sin z$ is continuous in the rectangular box $R = [0,\pi/2] \times [0,\pi/2] \times [0,\pi/2]$. It follows from Fubini's theorem that

$$\iiint_R f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

exists, and is equal to

$$\begin{aligned}&\int_0^{\pi/2} \left(\int_0^{\pi/2} \left(\int_0^{\pi/2} (\cos x - \sin y - \sin z)\,\mathrm{d}z\right)\mathrm{d}y\right)\mathrm{d}x = \int_0^{\pi/2} \left(\int_0^{\pi/2} \left(\frac{\pi}{2}\cos x - \frac{\pi}{2}\sin y - 1\right)\mathrm{d}y\right)\mathrm{d}x \\ &\quad = \int_0^{\pi/2} \left(\frac{\pi^2}{4}\cos x - \frac{\pi}{2} - \frac{\pi}{2}\right)\mathrm{d}x = \int_0^{\pi/2} \left(\frac{\pi^2}{4}\cos x - \pi\right)\mathrm{d}x = \frac{\pi^2}{4} - \frac{\pi^2}{2} = -\frac{\pi^2}{4}.\end{aligned}$$

EXAMPLE 5.7.2. We have

$$\int_0^1 \left(\int_0^3 \left(\int_0^2 (x+y)z\,\mathrm{d}z\right)\mathrm{d}y\right)\mathrm{d}x = \int_0^1 \left(\int_0^3 2(x+y)\,\mathrm{d}y\right)\mathrm{d}x = \int_0^1 (6x+9)\,\mathrm{d}x = 12.$$

We can also extend the integral to elementary regions. Instead of giving the definitions of a number of different types of elementary regions, we shall simply illustrate the technique with two examples. Our discussion here closely follows the ideas introduced in Section 5.4.

EXAMPLE 5.7.3. Consider the set

$$S = \{(x,y,z) \in \mathbb{R}^3 : x,y,z \geq 0 \text{ and } x^2+y^2+z^2 \leq 1\}$$

in $\mathbb{R}^3$. We shall find its volume by evaluating the triple integral

$$V(S) = \iiint_S \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

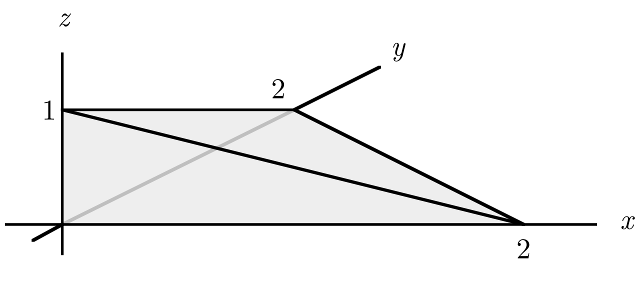
The region S can be described by

$$S = \{(x,y,z) \in \mathbb{R}^3 : x \in [0,1],\ y \in [0,\sqrt{1-x^2}] \text{ and } z \in [0,\sqrt{1-x^2-y^2}]\}.$$

Hence

$$\begin{aligned} V(S) &= \int_0^1 \left(\int_0^{\sqrt{1-x^2}} \left(\int_0^{\sqrt{1-x^2-y^2}} \mathrm{d}z\right)\mathrm{d}y\right)\mathrm{d}x = \int_0^1 \left(\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,\mathrm{d}y\right)\mathrm{d}x \\ &= \int_0^1 \left[\frac{1}{2}\left(y\sqrt{1-x^2-y^2} + (1-x^2)\sin^{-1}\frac{y}{\sqrt{1-x^2}}\right)\right]_0^{\sqrt{1-x^2}} \mathrm{d}x \\ &= \frac{\pi}{4}\int_0^1 (1-x^2)\,\mathrm{d}x = \frac{\pi}{6}. \end{aligned}$$

EXAMPLE 5.7.4. Consider the region S bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+2z=2$, with vertices $(0,0,0)$, $(2,0,0)$, $(0,2,0)$ and $(0,0,1)$.



Consider also the function $f(x,y,z) = 4xy+8z$. We can write

$$S = \left\{(x,y,z) \in \mathbb{R}^3 : x \in [0,2],\ y \in [0,2-x] \text{ and } z \in \left[0, \frac{2-x-y}{2}\right]\right\}.$$

Then

$$\begin{aligned}
&\int_0^2 \left(\int_0^{2-x} \left(\int_0^{(2-x-y)/2} (4xy + 8z)\,\mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x \\
&\quad = \int_0^2 \left(\int_0^{2-x} (2xy(2-x-y) + (2-x-y)^2)\,\mathrm{d}y \right) \mathrm{d}x \\
&\quad = \int_0^2 \left(\int_0^{2-x} (6xy - 2x^2y - 2xy^2 + 4 + x^2 + y^2 - 4x - 4y)\,\mathrm{d}y \right) \mathrm{d}x \\
&\quad = \int_0^2 \left(3x(2-x)^2 - x^2(2-x)^2 - \frac{2}{3}x(2-x)^3 + 4(2-x) + x^2(2-x) \right. \\
&\qquad\quad \left. + \frac{1}{3}(2-x)^3 - 4x(2-x) - 2(2-x)^2 \right) \mathrm{d}x \\
&\quad = \int_0^2 \left(\frac{8}{3} - \frac{4}{3}x - 2x^2 + \frac{5}{3}x^3 - \frac{1}{3}x^4 \right) \mathrm{d}x = \frac{28}{15}.
\end{aligned}$$

We can also write

$$S = \{(x,y,z) \in \mathbb{R}^3 : z \in [0,1],\ x \in [0, 2-2z] \text{ and } y \in [0, 2-2z-x]\}.$$

Then

$$\begin{aligned}
&\int_0^1 \left(\int_0^{2-2z} \left(\int_0^{2-2z-x} (4xy + 8z)\,\mathrm{d}y \right) \mathrm{d}x \right) \mathrm{d}z \\
&\quad = \int_0^1 \left(\int_0^{2-2z} (2x(2-2z-x)^2 + 8z(2-2z-x))\,\mathrm{d}x \right) \mathrm{d}z \\
&\quad = \int_0^1 \left(\int_0^{2-2z} (8x + 8xz^2 + 2x^3 - 24xz - 8x^2 + 8x^2z + 16z - 16z^2)\,\mathrm{d}x \right) \mathrm{d}z \\
&\quad = \int_0^1 \left(\frac{8}{3} + \frac{16}{3}z - 16z^2 + \frac{16}{3}z^3 + \frac{8}{3}z^4 \right) \mathrm{d}z = \frac{28}{15}.
\end{aligned}$$

Problems for Chapter 5

1. Evaluate each of the following integrals, where $R = [0,1] \times [0,1]$:

   a) $\displaystyle\iint_R (x^2 + y^3)\,\mathrm{d}x\mathrm{d}y$ b) $\displaystyle\iint_R \log((x+1)(y+1))\,\mathrm{d}x\mathrm{d}y$

2. Evaluate the integral $\displaystyle\int_{-1}^1 \left(\int_{-1}^1 \mathrm{e}^y y^2 \sin 2xy\,\mathrm{d}y \right) \mathrm{d}x.$

3. Suppose that the function f is continuous in the interval $[A,B]$, and the function g is continuous in the interval $[C,D]$. Suppose further that $R = [A,B] \times [C,D]$. Explain why the integral

   $$\iint_R f(x)g(y)\,\mathrm{d}x\mathrm{d}y$$

   exists and is equal to

   $$\left(\int_A^B f(x)\,\mathrm{d}x \right) \left(\int_C^D g(y)\,\mathrm{d}y \right).$$

4. Suppose that the function f is continuous in a rectangle $R = [A, B] \times [C, D]$, and $f(x, y) \geq 0$ for every $(x, y) \in R$. Show that
$$\iint_R f(x, y)\,\mathrm{d}x\mathrm{d}y = 0$$
if and only if $f(x, y) = 0$ for every $(x, y) \in R$.

5. Sketch the region S on the xy-plane bounded by the lines $x = 2$, $y = 1$ and the parabola $y = x^2$, and evaluate the double integral
$$\iint_S (x^3 + y^2)\,\mathrm{d}x\mathrm{d}y.$$

6. Consider the function $f(x, y) = \dfrac{x - y}{(x + y)^3}$, defined for every $(x, y) \in R = [0, 1] \times [0, 1]$, except at the point $(x, y) = (0, 0)$.
   a) Show that $\displaystyle\int_0^1 \left(\int_0^1 f(x, y)\,\mathrm{d}y \right) \mathrm{d}x = \frac{1}{2}$, and deduce that $\displaystyle\int_0^1 \left(\int_0^1 f(x, y)\,\mathrm{d}x \right) \mathrm{d}y = -\frac{1}{2}$.
   b) Show that $f(x, y)$ does not have a limit as $(x, y) \to (0, 0)$.
   c) Does the double integral $\displaystyle\iint_R f(x, y)\,\mathrm{d}x\mathrm{d}y$ exist as a Riemann integral? Give your reason(s).
   d) Comment on the results.

7. Consider the integral $\displaystyle\int_0^3 \left(\int_1^{\sqrt{4-y}} (x + y)\,\mathrm{d}x \right) \mathrm{d}y$.
   a) Sketch the area S on the xy-plane such that the integral is equal to $\displaystyle\iint_S (x + y)\,\mathrm{d}x\mathrm{d}y$.
   b) Interchange the order of integration, and evaluate the integral.

8. Show that the volume of the region bounded by $z = x^2 + y^2$, $z = 0$, $x = -a$, $x = a$, $y = -a$ and $y = a$, where $a > 0$, is given by $8a^4/3$.

9. Find each of the following integrals:
   a) $\displaystyle\int_0^1 \int_0^1 \int_0^1 (x + y + z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$
   b) $\displaystyle\int_0^1 \int_0^1 \int_0^1 (x + y + z)^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$

10. Let S be the region in $\mathbb{R}^3$ bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$, with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Show that
$$\iiint_S z\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{24}.$$

11. Let S be a "pyramid" with top vertex $(0, 0, 1)$ and base vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(1, 1, 0)$. Show that
$$\iiint_S (1 - z^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{3}{10}.$$
[Hint: Cut S by the plane $x = y$.]

12. Consider the region $S \subseteq \mathbb{R}^3$ bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = a$, where $a > 0$ is fixed.
    a) Make a rough sketch of the region S.
    b) Show that $\displaystyle\iiint_S (x^2 + y^2 + z^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{a^5}{20}$.

13. Show that $\displaystyle\int_0^1 \left(\int_0^1 \left(\int_{\sqrt{x^2+y^2}}^2 xyz\,\mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x = \frac{3}{8}$.