MULTIVARIABLE AND VECTOR ANALYSIS

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Chapter 6

CHANGE OF VARIABLES

6.1. Introduction

We consider double integrals of the type

$$\iint_S f(x,y)\,\mathrm{d}x\mathrm{d}y,$$

where $f : S \to \mathbb{R}$ is continuous in the region $S \subseteq \mathbb{R}^2$. Suppose that we wish to make a substitution $x = x(u,v)$ and $y = y(u,v)$. Then as in the theory of real valued functions of one real variable, we expect to have

$$\iint_S f(x,y)\,\mathrm{d}x\mathrm{d}y = \iint_{S^*} f(x(u,v),y(u,v))J(u,v)\,\mathrm{d}u\mathrm{d}v, \tag{1}$$

where $J(u,v)$ is a term arising from the substitution, and where S^* is a region in the uv-plane corresponding to the region S in the xy-plane.

Before we go any further, let us consider the following simple example in the theory of real valued functions of one real variable.

Example 6.1.1. To evaluate the integral

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x,$$

we may use the substitution $x = \cos\theta$. Then $\mathrm{d}x = -\sin\theta\,\mathrm{d}\theta$, and $\theta = \pi/2$ when $x = 0$ and $\theta = 0$ when $x = 1$. Hence

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \int_{\pi/2}^0 \sqrt{1-\cos^2\theta}(-\sin\theta)\,\mathrm{d}\theta = \int_0^{\pi/2} \sin^2\theta\,\mathrm{d}\theta = \ldots = \frac{\pi}{4}.$$

Clearly there is great potential for making a mistake by omitting a minus sign somewhere in the argument. We may instead proceed in the following way. Consider the function $T : [0, \pi/2] \to [0, 1]$, given by $T(\theta) = \cos\theta$ for every $\theta \in [0, \pi/2]$. Clearly this function is one-to-one and onto. We may well suspect that for any continuous function $f : [0, 1] \to \mathbb{R}$, we have

$$\int_{[0,1]} f(x)\,\mathrm{d}x = \int_{[0,\pi/2]} f(T(\theta)) \left|\frac{\mathrm{d}T}{\mathrm{d}\theta}\right| \mathrm{d}\theta.$$

Indeed, this is true when $f(x) = \sqrt{1-x^2}$ for every $x \in [0, 1]$, and is in fact true for every continuous function $f : [0, 1] \to \mathbb{R}$.

6.2. Planar Transformations

In this section, we shall study functions of the type $T : S^* \to S$, where S^* and S are sets in $\mathbb{R}^2$.

Suppose that $S^* \subseteq \mathbb{R}^2$ is given. Consider a function $T : S^* \to \mathbb{R}^2$. Then for any $(u, v) \in S^*$, the point $T(u, v)$ is called the image of (u, v) inder T, and the set

$$T(S^*) = \{T(u, v) : (u, v) \in S^*\}$$

is called the range of the function T.

Example 6.2.1. Suppose that $S^* = [0, 1] \times [0, 2\pi]$. The function $T : S^* \to \mathbb{R}^2 : (r, \theta) \mapsto (r\cos\theta, r\sin\theta)$ has range

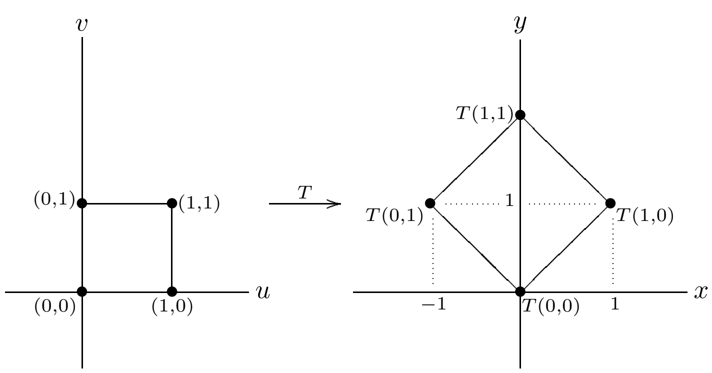
$$T(S^*) = \{(r\cos\theta, r\sin\theta) : (r, \theta) \in [0, 1] \times [0, 2\pi]\},$$

the closed disc in $\mathbb{R}^2$ with radius 1 and centre $(0, 0)$.

Example 6.2.2. Suppose that $S^* = [0, 1] \times [0, 1]$. Consider the function $T : S^* \to \mathbb{R}^2$, given for every $(u, v) \in S^*$ by $T(u, v) = (x, y)$, where

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \qquad \text{and} \qquad x + \mathrm{i}y = (1 + \mathrm{i})(u + \mathrm{i}v).$$

Clearly $T(S^*)$ is a square with vertices $(0, 0)$, $(0, 2)$ and $(\pm 1, 1)$.



Definition. A function $T : S^* \to \mathbb{R}^2$ is said to be one-to-one if for every $(u', v'), (u'', v'') \in S^*$, we have $(u', v') = (u'', v'')$ whenever $T(u', v') = T(u'', v'')$. In other words, no two distinct $(u', v'), (u'', v'') \in S^*$ can have the same image.

EXAMPLE 6.2.3. In Example 6.2.1, the function $T : S^* \to \mathbb{R}^2$ is not one-to-one. It is easy to check that $T(0,0) = T(0,\pi) = 0$.

EXAMPLE 6.2.4. In Example 6.2.2, it is easily seen that the function $T : S^* \to \mathbb{R}^2$ is a rotation about the point $(0,0)$ through an angle $\pi/4$, together with a magnification by $\sqrt{2}$ about the point $(0,0)$. It is clear that if $T(u',v') = (x',y')$, $T(u'',v'') = (x'',y'')$ and $(x',y') = (x'',y'')$, then since the matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is invertible, we must have $(u',v') = (u'',v'')$. Hence $T : S^* \to \mathbb{R}^2$ is one-to-one.

EXAMPLE 6.2.5. Let us vary Example 6.2.1, and consider the rectangle $S^* = (0,1] \times [0,2\pi)$. Then the function $T : S^* \to \mathbb{R}^2 : (r,\theta) \mapsto (r\cos\theta, r\sin\theta)$ is one-to-one, with range

$$T(S^*) = \{(r\cos\theta, r\sin\theta) : (r,\theta) \in (0,1] \times [0,2\pi)\},$$

the closed disc in $\mathbb{R}^2$ with radius 1 and centre $(0,0)$, punctured at the point $(0,0)$.

In practice, our problem is the following: Given a region $S \subseteq \mathbb{R}^2$, we need to find a region $S^* \subseteq \mathbb{R}^2$ and a one-to-one function $T : S^* \to \mathbb{R}^2$ such that $T(S^*) = S$. It is therefore convenient to consider functions of the type $T : S^* \to S$ instead of $T : S^* \to \mathbb{R}^2$. Note that this change does not affect our definition of one-to-one functions given earlier, and we can rephrase it as follows.

DEFINITION. A function $T : S^* \to S$ is said to be one-to-one if for every $(u',v'), (u'',v'') \in S^*$, we have $(u',v') = (u'',v'')$ whenever $T(u',v') = T(u'',v'')$.

To characterize the property $T(S^*) = S$, we have the following definition.

DEFINITION. A function $T : S^* \to S$ is said to be onto if for every $(x,y) \in S$, there exists $(u,v) \in S^*$ such that $T(u,v) = (x,y)$.

EXAMPLE 6.2.6. Suppose that $S^* = [0,1] \times [0,2\pi]$ and $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$. Then the function $T : S^* \to \mathbb{R}^2 : (r,\theta) \mapsto (r\cos\theta, r\sin\theta)$ is onto.

EXAMPLE 6.2.7. In Example 6.2.2, if we change the codomain of the function T to the closed rectangle S with vertices $(0,0)$, $(0,2)$ and $(\pm 1, 1)$, then $T : S^* \to S$ is one-to-one and onto.

EXAMPLE 6.2.8. Suppose that $S^* = (0,1] \times [0,2\pi)$ and $S = \{(x,y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \le 1\}$. Then the function $T : S^* \to \mathbb{R}^2 : (r,\theta) \mapsto (r\cos\theta, r\sin\theta)$ is one-to-one and onto.

6.3. The Jacobian

In this section, we shall turn our attention to the term $J(u,v)$ in the expression (1). We shall begin by studying the special case when $f(x,y) = 1$ for every $(x,y) \in S$, so that the expression (1) is of the form

$$\iint_S \mathrm{d}x\mathrm{d}y = \iint_{S^*} J(u,v)\,\mathrm{d}u\mathrm{d}v. \tag{2}$$

Note that the left hand side of (2) represents the area of S.

EXAMPLE 6.3.1. We study rectangular coordinates (x,y) and polar coordinates (r,θ), related by

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$

Consider the unit disc

$$S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}.$$

Let

$$S^* = \{(r,\theta) \in \mathbb{R}^2 : 0 \le r \le 1 \text{ and } 0 \le \theta \le 2\pi\}.$$

Then the function

$$T : S^* \to \mathbb{R}^2 : (r,\theta) \mapsto (r\cos\theta, r\sin\theta)$$

is one-to-one and onto, provided that we remove the lines $r = 0$ and $\theta = 2\pi$ from S^* and the point $(x,y) = (0,0)$ from S. Clearly

$$\iint_S \mathrm{d}x\mathrm{d}y = \pi.$$

On the other hand,

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{pmatrix} = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r.$$

By Fubini's theorem, we have

$$\iint_{S^*} \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| \mathrm{d}r\mathrm{d}\theta = \iint_{S^*} r\,\mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi} \left(\int_0^1 r\,\mathrm{d}r\right) \mathrm{d}\theta = \pi.$$

Here we detect a small problem. The function $T : S^* \to S$ may not be one-to-one and onto, unless we remove some subsets of S^* and S. Let us first of all consider such exceptional subsets.

DEFINITION. Suppose that $S \subseteq \mathbb{R}^2$. Then we say that a subset $E \subseteq S$ is exceptional if E is contained in a finite union of curves of type 1 or 2 in S.

DEFINITION. Suppose that S^* and S are sets in $\mathbb{R}^2$. Then we say that a function $T : S^* \to S$ is essentially one-to-one and onto if there exist exceptional subsets $E^* \subseteq S^*$ and $E \subseteq S$ such that the function $T : S^* \setminus E^* \to S \setminus E$ is one-to-one and onto.

EXAMPLE 6.3.2. In Example 6.3.1, we can take

$$E^* = \{(r,\theta) \in S^* : r = 0 \text{ or } \theta = 2\pi\} \qquad \text{and} \qquad E = \{(0,0)\}.$$

Then

$$S^* \setminus E^* = \{(r,\theta) \in \mathbb{R}^2 : 0 < r \le 1 \text{ and } 0 \le \theta < 2\pi\} \qquad \text{and} \qquad S \setminus E = \{(x,y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \le 1\}.$$

We have shown in Example 6.2.8 that the function

$$T : S^* \setminus E^* \to S \setminus E : (r,\theta) \mapsto (r\cos\theta, r\sin\theta)$$

is one-to-one and onto.

THEOREM 6A. *Suppose that $S^* \subseteq \mathbb{R}^2$ and $S \subseteq \mathbb{R}^2$ are elementary regions, and that the function $T : S^* \to S$, where $T(u,v) = (x(u,v), y(u,v))$ for every $(u,v) \in S^*$, has continuous partial derivatives. Suppose further that $T : S^* \to S$ is essentially one-to-one and onto. Then for any Riemann integrable function $f : S \to \mathbb{R}$, we have*

$$\iint_S f(x,y)\,\mathrm{d}x\mathrm{d}y = \iint_{S^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v,$$

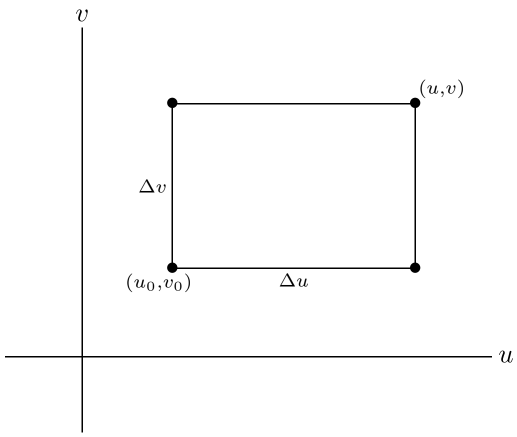
where the Jacobian determinant of T is given by

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix}.$$

REMARK. An alternative approach is to restrict S^* to open sets and take $S = T(S^*)$. However, the problem with this approach is that since S is usually given, it may not always be possible to find a suitable open set S^* such that $T(S^*) = S$ precisely, and we face the same difficulty, albeit in a slightly different disguise.

We shall not prove Theorem 6A. Instead we shall only briefly indicate the ideas. Consider a small rectangle

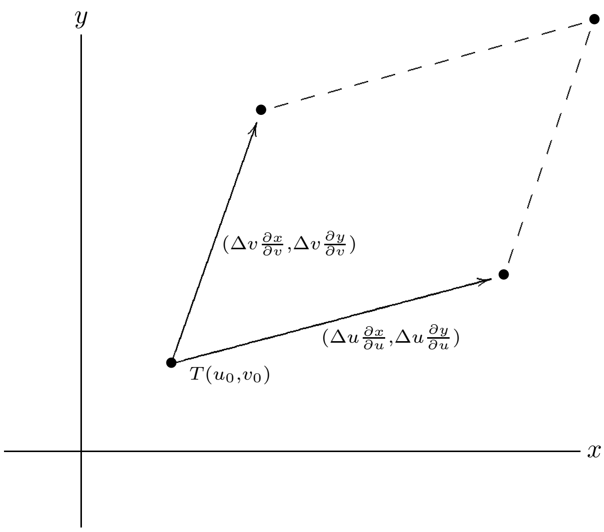
(3)



in S^* with bottom left vertex (u_0, v_0) and top right vertex $(u, v) = (u_0 + \Delta u, v_0 + \Delta v)$. We know from our discussion of differentiability in Chapter 2 that a good approximation to $T(u, v)$ is given by the linear mapping

(4) $$T(u_0, v_0) + (\mathbf{D}T)(u_0, v_0)\begin{pmatrix} u - u_0 \\ v - v_0 \end{pmatrix} = T(u_0, v_0) + (\mathbf{D}T)(u_0, v_0)\begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix},$$

where $(\mathbf{D}T)(u_0, v_0)$ denotes the total derivative of T at (u_0, v_0). The image of the rectangle (3) under the mapping (4) is a parallelogram



with one vertex at $T(u_0,v_0)$ and adjacent sides, corresponding to Δu and Δv, given by the vectors

$$(\mathbf{D}T)(u_0,v_0)\begin{pmatrix}\Delta u\\ 0\end{pmatrix}=\begin{pmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{pmatrix}\begin{pmatrix}\Delta u\\ 0\end{pmatrix}=\Delta u\begin{pmatrix}\dfrac{\partial x}{\partial u}\\ \dfrac{\partial y}{\partial u}\end{pmatrix}$$

and

$$(\mathbf{D}T)(u_0,v_0)\begin{pmatrix}0\\ \Delta v\end{pmatrix}=\begin{pmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{pmatrix}\begin{pmatrix}0\\ \Delta v\end{pmatrix}=\Delta v\begin{pmatrix}\dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial v}\end{pmatrix}$$

respectively (here the partial derivatives are supposed to be evaluated at (u_0,v_0)). Clearly the area of this parallelogram is

$$\left|\det\begin{pmatrix}\Delta u\dfrac{\partial x}{\partial u} & \Delta v\dfrac{\partial x}{\partial v}\\ \Delta u\dfrac{\partial y}{\partial u} & \Delta v\dfrac{\partial y}{\partial v}\end{pmatrix}\right|=\left|\det\begin{pmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{pmatrix}\right|\Delta u\Delta v=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u\Delta v.$$

EXAMPLE 6.3.3. Suppose that $0\le\theta_1<\theta_2\le 2\pi$. Consider the region

$$S=\{(r\cos\theta,r\sin\theta):\theta_1\le\theta\le\theta_2\text{ and }g_1(\theta)\le r\le g_2(\theta)\},$$

where $g_1(\theta)\le g_2(\theta)$ for every $\theta\in[0,2\pi]$. Writing $(x,y)=(r\cos\theta,r\sin\theta)$, we can calculate the area of S as

$$A(S)=\iint_S \mathrm{d}x\mathrm{d}y=\iint_{S^*}\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|\mathrm{d}r\mathrm{d}\theta,$$

where

$$S^*=\{(r,\theta)\in\mathbb{R}^2:\theta_1\le\theta\le\theta_2\text{ and }g_1(\theta)\le r\le g_2(\theta)\}.$$

Then

$$A(S)=\int_{\theta_1}^{\theta_2}\left(\int_{g_1(\theta)}^{g_2(\theta)}\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|\mathrm{d}r\right)\mathrm{d}\theta=\int_{\theta_1}^{\theta_2}\left(\int_{g_1(\theta)}^{g_2(\theta)}r\,\mathrm{d}r\right)\mathrm{d}\theta=\int_{\theta_1}^{\theta_2}\frac{g_2^2(\theta)-g_1^2(\theta)}{2}\,\mathrm{d}\theta.$$

EXAMPLE 6.3.4. Consider the region

$$S=\{(x,y)\in\mathbb{R}^2:x,y\ge 0\text{ and }1\le x^2+y^2\le 4\}$$

in the first quadrant between the two circles of radii 1 and 2 centred at $(0,0)$. We showed in Example 5.4.4 that

$$\iint_S(x^2+y^2)\,\mathrm{d}x\mathrm{d}y=\frac{15\pi}{8}.$$

We now use the substitution $x = r\cos\theta$ and $y = r\sin\theta$ instead, and consider the rectangle

$$S^* = \{(r,\theta) \in \mathbb{R}^2 : 1 \le r \le 2 \text{ and } 0 \le \theta \le \pi/2\}.$$

Note that the function

$$T : S^* \to S : (r,\theta) \mapsto (r\cos\theta, r\sin\theta)$$

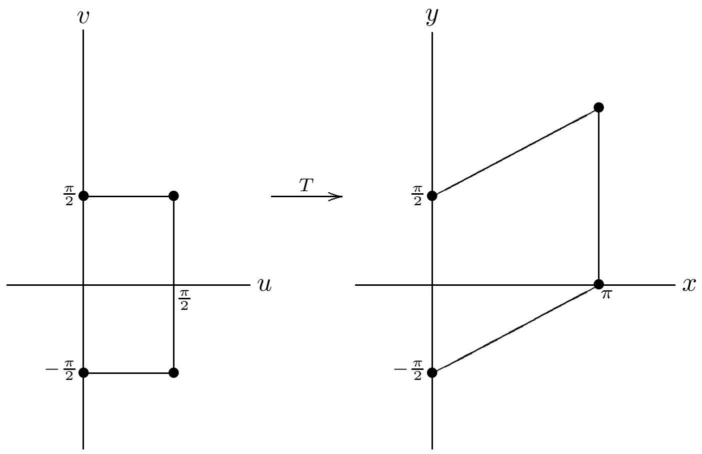
is one-to-one and onto. By Theorem 6A, we have

$$\iint_S (x^2+y^2)\,\mathrm{d}x\mathrm{d}y = \iint_{S^*} (r^2\cos^2\theta + r^2\sin^2\theta) \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| \mathrm{d}r\mathrm{d}\theta = \iint_{S^*} r^3\,\mathrm{d}r\mathrm{d}\theta = \ldots = \frac{15\pi}{8}.$$

EXAMPLE 6.3.5. Consider the parallelogram S with vertices $(\pi,0)$, (π,π) and $(0,\pm\pi/2)$. We showed in Example 5.4.2 that

$$\iint_S \sin x\cos y\,\mathrm{d}x\mathrm{d}y = \frac{8}{3}.$$

We now use the substitution $x = 2u$ and $y = u+v$ instead, and consider the rectangle S^* with vertices $(0,\pm\pi/2)$ and $(\pi/2,\pm\pi/2)$.



Note that the function

$$T : S^* \to S : (u,v) \mapsto (2u, u+v)$$

is one-to-one and onto. By Theorem 6A, we have

$$\begin{aligned}\iint_S \sin x\cos y\,\mathrm{d}x\mathrm{d}y &= \iint_{S^*} \sin 2u\cos(u+v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v = 2\iint_{S^*} \sin 2u(\cos u\cos v - \sin u\sin v)\,\mathrm{d}u\mathrm{d}v \\ &= 2\left(\int_0^{\pi/2} \sin 2u\cos u\,\mathrm{d}u\right)\left(\int_{-\pi/2}^{\pi/2} \cos v\,\mathrm{d}v\right) - 2\left(\int_0^{\pi/2} \sin 2u\sin u\,\mathrm{d}u\right)\left(\int_{-\pi/2}^{\pi/2} \sin v\,\mathrm{d}v\right) \\ &= 2\left(\int_0^{\pi/2} \sin 2u\cos u\,\mathrm{d}u\right)\left(\int_{-\pi/2}^{\pi/2} \cos v\,\mathrm{d}v\right) = \ldots = \frac{8}{3}.\end{aligned}$$

EXAMPLE 6.3.6. Consider the repeated integral

$$\int_0^1 \left(\int_0^1 (x^2+y^2)^{3/2}\,\mathrm{d}x\right)\mathrm{d}y = \iint_R (x^2+y^2)^{3/2}\,\mathrm{d}x\mathrm{d}y,$$

where $R = [0,1] \times [0,1]$. By splitting R into two triangles along the line $y = x$, it is easy to see that

$$\begin{aligned}\iint_R (x^2+y^2)^{3/2}\,\mathrm{d}x\mathrm{d}y &= \int_0^1 \left(\int_0^x (x^2+y^2)^{3/2}\,\mathrm{d}y\right)\mathrm{d}x + \int_0^1 \left(\int_0^y (x^2+y^2)^{3/2}\,\mathrm{d}x\right)\mathrm{d}y \\ &= 2\iint_S (x^2+y^2)^{3/2}\,\mathrm{d}x\mathrm{d}y,\end{aligned}$$

where S is the triangular region with vertices $(0,0)$, $(1,0)$ and $(1,1)$. We use the substitution $x = r\cos\theta$ and $y = r\sin\theta$, and consider the region

$$S^* = \{(r,\theta) \in \mathbb{R}^2 : 0 \le \theta \le \pi/4 \text{ and } 0 \le r \le \sec\theta\}.$$

Note that the function

$$T : S^* \to S : (r,\theta) \mapsto (r\cos\theta, r\sin\theta)$$

is one-to-one and onto, provided that we remove the line $r = 0$ from S^* and the point $(x,y) = (0,0)$ from S. By Theorem 6A, we have

$$\begin{aligned}\iint_S (x^2+y^2)^{3/2}\,\mathrm{d}x\mathrm{d}y &= \iint_{S^*} (r^2\cos^2\theta + r^2\sin^2\theta)^{3/2}\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|\mathrm{d}r\mathrm{d}\theta = \int_0^{\pi/4}\left(\int_0^{\sec\theta} r^4\,\mathrm{d}r\right)\mathrm{d}\theta \\ &= \frac{1}{5}\int_0^{\pi/4}\sec^5\theta\,\mathrm{d}\theta = \left[\frac{1}{20}\tan\theta + \sec^3\theta + \frac{3}{40}\tan\theta\sec\theta + \frac{3}{40}\log|\tan\theta+\sec\theta|\right]_0^{\pi/4} \\ &= \frac{7\sqrt{2}}{40} + \frac{3}{40}\log(1+\sqrt{2}).\end{aligned}$$

6.4. Triple Integrals

Our discussion so far can be extended to triple integrals with no extra complication.

DEFINITION. Suppose that $S \subseteq \mathbb{R}^3$. Then we say that a subset $E \subseteq S$ is exceptional if E is contained in a finite union of surfaces in S representing one of the three variables as a continuous function of the other two variables.

DEFINITION. Suppose that S^* and S are sets in $\mathbb{R}^3$. Then we say that a function $T : S^* \to S$ is essentially one-to-one and onto if there exist exceptional subsets $E^* \subseteq S^*$ and $E \subseteq S$ such that the function $T : S^* \setminus E^* \to S \setminus E$ is one-to-one and onto.

THEOREM 6B. *Suppose that $S^* \subseteq \mathbb{R}^3$ and $S \subseteq \mathbb{R}^3$ are elementary regions, and that the function $T : S^* \to S$, where $T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$ for every $(u,v,w) \in S^*$, has continuous partial derivatives. Suppose further that $T : S^* \to S$ is essentially one-to-one and onto. Then for any Riemann integrable function $f : S \to \mathbb{R}$, we have*

$$\iiint_S f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{S^*} f(x(u,v,w), y(u,v,w), z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\mathrm{d}u\mathrm{d}v\mathrm{d}w,$$

where the Jacobian determinant of T is given by

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{pmatrix}.$$

REMARKS. (1) The image under T of a rectangular box in S^* with side lengths Δu, Δv and Δw can be approximated by a parallelepiped in S with volume

$$\left|\det\begin{pmatrix} \Delta u\dfrac{\partial x}{\partial u} & \Delta v\dfrac{\partial x}{\partial v} & \Delta w\dfrac{\partial x}{\partial w} \\ \Delta u\dfrac{\partial y}{\partial u} & \Delta v\dfrac{\partial y}{\partial v} & \Delta w\dfrac{\partial y}{\partial w} \\ \Delta u\dfrac{\partial z}{\partial u} & \Delta v\dfrac{\partial z}{\partial v} & \Delta w\dfrac{\partial z}{\partial w} \end{pmatrix}\right| = \left|\det\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{pmatrix}\right| \Delta u\Delta v\Delta w = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \Delta u\Delta v\Delta w.$$

(2) Cylindrical coordinates (r,θ,z) are related to rectangular coordinates (x,y,z) in $\mathbb{R}^3$ by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

Hence

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det\begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial z} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial z} \end{pmatrix} = \det\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = r,$$

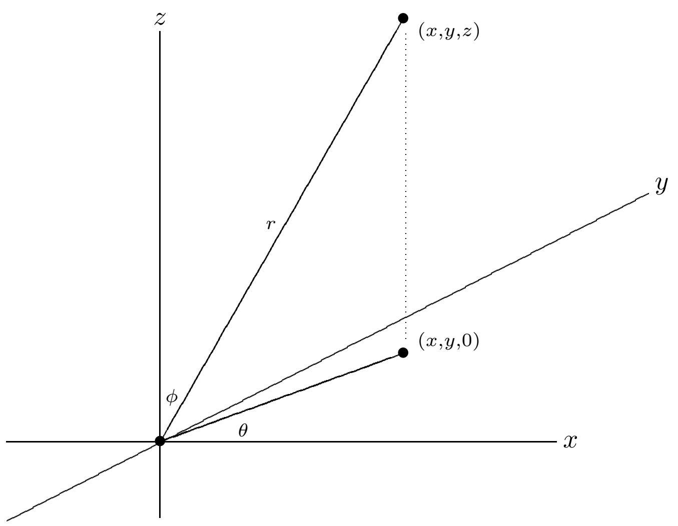
so that

$$\iiint_S f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{S^*} f(r\cos\theta, r\sin\theta, z) r\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}z.$$

Here, we essentially leave the variable z alone, and replace the variables x and y by the variables r and θ in the same way as with rectangular and polar coordinates in $\mathbb{R}^2$.

(3) Spherical coordinates (r,ϕ,θ) are related to rectangular coordinates (x,y,z) in $\mathbb{R}^3$ by

$$x = r\sin\phi\cos\theta, \qquad y = r\sin\phi\sin\theta, \qquad z = r\cos\phi.$$



We have

$$\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)} = \det\begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial \theta} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial \theta} \end{pmatrix} = \det\begin{pmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -r\sin\phi & 0 \end{pmatrix}$$

$$= (\cos\phi)\det\begin{pmatrix} r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ r\cos\phi\sin\theta & r\sin\phi\cos\theta \end{pmatrix} + (r\sin\phi)\det\begin{pmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta \end{pmatrix}$$

$$= (\cos\phi)(r^2\cos\phi\sin\phi\cos^2\theta + r^2\cos\phi\sin\phi\sin^2\theta) + (r\sin\phi)(r\sin^2\phi\cos^2\theta + r\sin^2\phi\sin^2\theta)$$

$$= r^2\sin\phi\cos^2\phi + r^2\sin\phi\sin^2\phi = r^2\sin\phi,$$

so that

$$\iiint_S f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{S^*} f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)r^2\sin\phi\,\mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta.$$

Example 6.4.1. Consider the integral

$$\iiint_S (x^2+y^2+z^2)^{1/2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

where

$$S = \{(x,y,z)\in\mathbb{R}^3 : y,z \geq 0 \text{ and } 4 \leq x^2+y^2+z^2 \leq 9\}.$$

We use spherical coordinates and the substitution

$$x = r\sin\phi\cos\theta, \qquad y = r\sin\phi\sin\theta, \qquad z = r\cos\phi,$$

and consider the set

$$S^* = \left\{(r,\phi,\theta)\in\mathbb{R}^3 : r\in[2,3],\ \phi\in\left[0,\frac{\pi}{2}\right] \text{ and } \theta\in[0,\pi]\right\}.$$

Note that the function

$$T : S^* \to S : (r,\phi,\theta) \mapsto (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)$$

is one-to-one and onto. By Theorem 6B and noting Remark (3) above, we have

$$\iiint_S (x^2+y^2+z^2)^{1/2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{S^*} (r^2\sin^2\phi\cos^2\theta + r^2\sin^2\phi\sin^2\theta + r^2\cos^2\phi)^{1/2} r^2\sin\phi\,\mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta$$

$$= \iiint_{S^*} r^3\sin\phi\,\mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta = \left(\int_2^3 r^3\,\mathrm{d}r\right)\left(\int_0^{\pi/2}\sin\phi\,\mathrm{d}\phi\right)\left(\int_0^{\pi}\mathrm{d}\theta\right) = \frac{65\pi}{4}.$$

EXAMPLE 6.4.2. Consider the integral

$$\iiint_S (x^2+y^2+z^2)^{5/2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

where

$$S = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2 \le 1\}.$$

We use spherical coordinates and the substitution

$$x = r\sin\phi\cos\theta, \qquad y = r\sin\phi\sin\theta, \qquad z = r\cos\phi,$$

and consider the set

$$S^* = \{(r,\phi,\theta)\in\mathbb{R}^3 : r\in[0,1],\ \phi\in[0,\pi] \text{ and } \theta\in[0,2\pi]\}.$$

Note that the function

$$T : S^* \to S : (r,\phi,\theta) \mapsto (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)$$

is one-to-one and onto, provided that we remove the planes $r=0$ and $\theta=2\pi$ from S^* and the point $(0,0,0)$ from S. By Theorem 6B and noting Remark (3) above, we have

$$\iiint_S (x^2+y^2+z^2)^{5/2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{S^*} r^7\sin\phi\,\mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta = \left(\int_0^1 r^7\,\mathrm{d}r\right)\left(\int_0^\pi \sin\phi\,\mathrm{d}\phi\right)\left(\int_0^{2\pi}\mathrm{d}\theta\right) = \frac{\pi}{2}.$$

EXAMPLE 6.4.3. Consider the integral

$$\iiint_S (x^2+y^2)^{1/2} z\,\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

where

$$S = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2 \le 1 \text{ and } 0\le z\le 1\}.$$

We use cylindrical coordinates and the substitution

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$

and consider the set

$$S^* = \{(r,\theta,z)\in\mathbb{R}^3 : r\in[0,1],\ \theta\in[0,2\pi] \text{ and } z\in[0,1]\}.$$

Note that the function

$$T : S^* \to S : (r,\theta,z) \mapsto (r\cos\theta, r\sin\theta, z)$$

is one-to-one and onto, provided that we remove the planes $r=0$ and $\theta=2\pi$ from S^* and the point $(0,0,0)$ from S. By Theorem 6B and noting Remark (2) above, we have

$$\begin{aligned}\iiint_S (x^2+y^2)^{1/2} z\,\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \iiint_{S^*} (r^2\cos^2\theta + r^2\sin^2\theta)^{1/2} zr\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}z \\ &= \left(\int_0^1 r^2\,\mathrm{d}r\right)\left(\int_0^{2\pi}\mathrm{d}\theta\right)\left(\int_0^1 z\,\mathrm{d}z\right) = \frac{\pi}{3}.\end{aligned}$$

Problems for Chapter 6

1. By changing to polar coordinates, evaluate the integral $\displaystyle\int_0^1 \left(\int_0^{\sqrt{1-x^2}} \mathrm{e}^{-x^2-y^2}\,\mathrm{d}y\right)\mathrm{d}x$.

2. By writing $2x = r - s$ and $2y = r + s$, evaluate the integral $\displaystyle\int_0^{1/2} \left(\int_x^{1-x} \left(\frac{x-y}{x+y}\right)^2 \mathrm{d}y\right)\mathrm{d}x$.

3. By using the transformation $x + y = u$ and $y = uv$, show that $\displaystyle\int_0^1 \left(\int_0^{1-x} \mathrm{e}^{y/(x+y)}\,\mathrm{d}y\right)\mathrm{d}x = \frac{\mathrm{e}-1}{2}$.

   [Remark: Take great care when calculating the Jacobian.]

4. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ are differentiable. Suppose further that $(u, v) = f(s, t)$ and $(x, y) = g(u, v)$, so that $(x, y) = (g \circ f)(s, t)$.
   a) Use the Chain rule $(\mathbf{D}(g \circ f))(s, t) = (\mathbf{D}g)(u, v)(\mathbf{D}f)(s, t)$ to show that
   $$\frac{\partial(x, y)}{\partial(s, t)} = \frac{\partial(x, y)}{\partial(u, v)}\frac{\partial(u, v)}{\partial(s, t)}.$$
   b) Show that $\displaystyle\frac{\partial(x, y)}{\partial(u, v)}\frac{\partial(u, v)}{\partial(x, y)} = 1$ provided that $\displaystyle\frac{\partial(x, y)}{\partial(u, v)} \neq 0$.

5. Let S be the region in the first quadrant of $\mathbb{R}^2$ bounded by $x^2 - y^2 = 1$, $x^2 - y^2 = 9$, $xy = 2$ and $xy = 4$. Consider also the substitution $u = x^2 - y^2$ and $v = 2xy$.
   a) Draw a picture of the region S.
   b) Use the result in Question 4(b) to find $\partial(x, y)/\partial(u, v)$.
   c) Show that $2x^2 = (u^2 + v^2)^{1/2} + u$ and $2y^2 = (u^2 + v^2)^{1/2} - u$.
   d) Let $S^* = \{(u, v) \in \mathbb{R}^2 : u \in [1, 9] \text{ and } v \in [4, 8]\}$. Show that $T : S^* \to S : (u, v) \mapsto (x, y)$ is one-to-one and onto.
   e) By changing variables, show that $\displaystyle\iint_S (x^2 + y^2)\,\mathrm{d}x\mathrm{d}y = 8$.

6. Evaluate each of the following triple integrals by using a suitable substitution:
   a) $\displaystyle\iiint_S (x^2 + y^2 + z^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, where $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \le 4 \text{ and } 0 \le z \le 2\}$
   b) $\displaystyle\iiint_S (x^2 + y^2 + z^2)^{1/2}\mathrm{e}^{-(x^2+y^2+z^2)}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, where $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le 4\}$
   c) $\displaystyle\iiint_S \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{(x^2 + y^2 + z^2)^{3/2}}$, where $S = \{(x, y, z) \in \mathbb{R}^3 : a^2 \le x^2 + y^2 + z^2 \le b^2\}$ with $b > a > 0$

7. Let S be the region in $\mathbb{R}^3$ bounded by the paraboloid $z = x^2 + y^2$ and the cone $z = \sqrt{x^2 + y^2}$.
   a) Find the set of points $(x, y, z) \in \mathbb{R}^3$ where the paraboloid intersects the cone.
   b) Draw the cross section of the region S on the plane $y = 0$.
   c) Show that the volume of S is equal to
   $$\iint_T (\sqrt{x^2 + y^2} - (x^2 + y^2))\,\mathrm{d}x\mathrm{d}y,$$
   where $T = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$.
   d) By changing to polar coordinates, show that the volume of S is equal to $\pi/6$.

8. By considering a suitable set of coordinates in $\mathbb{R}^3$ and making the necessary change of variables, show that

$$\iiint_S \sqrt{x^2+y^2+z^2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{27\pi(2\sqrt{2}-1)}{2},$$

where S is the region in $\mathbb{R}^3$ bounded by the plane $z=3$ and the cone $z=\sqrt{x^2+y^2}$.