# Optimization in several variables



## 1. Optimization without constraints

Given a function of two variables, $z = f(x, y)$, it is often useful to find the points,

$$(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2),\ \ldots$$

where the function attains **optimal** values, i.e., **relative minimum** or **relative maximum values**. A $f(x, y)$ has a relative maximum [minimum] value at the point $(x_0, y_0)$ if

$$f(x_0, y_0) \geq f(x, y) \qquad [f(x_0, y_0) \leq f(x, y)]$$

for all points $(x, y)$ that are **sufficiently close** to $(x_0, y_0)$. The value $z_0 = f(x_0, y_0)$ is an **absolute** maximum [minimum] value, if $z_0$ is greater than [less than] **all** other values of the function.

As in the case of functions of one variable, the first step to finding relative maxima and minima is to find **critical points** of the function in question. The point $(x_0, y_0)$ is a critical point of the function $f(x, y)$ if

$$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0.$$

**Example 1.1.**

Find the critical points of $f(x, y) = x^3 + 2xy - 2y^2 - 10x + 3$. To do this we solve the pair of equations

$$\begin{cases} f_x &= 3x^2 + 2y - 10 &= 0 \\ f_y &= 2x - 4y &= 0 \end{cases}$$

The second equation implies that $x = 2y$, and substituting this into the first equation gives the (quadratic) equation in $x$

$$3x^2 + x - 10 = 0,$$

which has two solutions $x_1 = -2$ and $x_2 = 5/3$. So the critical points in this case are $(x_1, y_1) = (-2, -1)$ and $(x_2, y_2) = (5/3, 5/6)$.

This principle applies when there are three, four or more variables as well:

**Fact 1.** *The (relative) extreme value(s) of the function*

$$f(x_1, x_2, \ldots, x_n)$$

*occur at the solution(s) of the system of equations*

$$\begin{aligned} f_{x_1}(x_1, x_2, \ldots, x_n) &= 0 \\ f_{x_2}(x_1, x_2, \ldots, x_n) &= 0 \\ &\vdots \\ f_{x_n}(x_1, x_2, \ldots, x_n) &= 0. \end{aligned}$$

**Example 1.2.**

Find the critical points of the function

$$f(x,y,z) = x^2 + 2y^2 + 5z^2 - 2xy - 4yz + 2x - 2y - 2z + 13.$$

We need to solve the system of equations given by setting the three first order derivatives of $f$ equal to 0:

$$\begin{aligned} f_x = 2x - 2y + 2 &= 0 \\ f_y = -2x + 4y - 4z - 2 &= 0 \\ f_z = -4y + 10z - 2 &= 0 \end{aligned}$$

From the first equation we see that $x = y - 1$, and from the third equation we see that $z = (1+2y)/5$. We substitute for $x$ and $z$ into the second equation, which gives

$$-2(y-1) + 4y - 4(1+2y)/5 - 2 = 0 \implies \frac{2}{5}y - \frac{4}{5} = 0 \implies y_0 = 2.$$

So $x_0 = 2 - 1 = 1$ and $z_0 = (1+4)/5 = 1$, and the (only) critical point is $(x_0, y_0, z_0) = (1, 2, 1)$.

## 2. The second derivative test

As in the case of functions of one variable, we need a method to determine whether the value of the function at the critical point is a (relative) maximum or minimum value. And, as in that case, there is a *second derivative test* that we can use. However, as the number of variables grows so does the complexity of the corresponding second derivative test, so I will only present the second derivative test for functions of two variables.

**Fact 2** (The second derivative test)**.** *Suppose that $(x_0, y_0)$ is a critical point of the function $f(x,y)$. To determine whether the critical point $f(x_0, y_0)$ is a relative minimum value, relative maximum value or neither, we compute the* **discriminant** *of $f$:*

$$D(x,y) = f_{xx}(x,y) \cdot f_{yy}(x,y) - [f_{xy}(x,y)]^2,$$

*and evaluate $D(x_0, y_0)$ and $f_{xx}(x_0, y_0)$. There are four cases:*

1. *If $D(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) > 0$ then $f(x_0,y_0)$ is a relative* **minimum** *value.*
2. *If $D(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) < 0$ then $f(x_0,y_0)$ is a relative* **maximum** *value.*
3. *If $D(x_0,y_0) < 0$ then the point $(x_0, y_0, f(x_0,y_0))$ is a* **saddle point***. In this case $f(x_0,y_0)$ is neither a minimum nor a maximum. It's called a saddle point because the graph often looks like a saddle around that point — going up in two directions and going down in two others.*
4. *If $D(x_0,y_0) = 0$ then the test does not give any information about the point — it could fall into any of the cases above.*

**Comment:** As with all second derivative tests, this test only classifies *relative* extreme values. If we need to determine whether a given value is an *absolute* extreme value (or not), then other considerations are necessary.

**Example 2.1.**

I'll apply the second derivative to the critical points that we found in Example 1.1 for the function $f(x,y) = x^3 + 2xy - 2y^2 - 10x + 3$. First we compute

$$f_{xx} = 6x, \quad f_{yy} = -4, \quad f_{xy} = 2 \quad \text{and} \quad D = -24x - 4.$$

Next, $D(5/3, 5/6) = -44 < 0$, so the point $(5/3, 5/6, -7.648)$ is a saddle point on the graph of $z = f(x,y)$. On the other hand, $D(-2,-1) = 44 > 0$ and $f_{xx}(-2,-1) = 48$, so $f(-2,-1) = 7$ is a *relative* minimum value. In this case, we can tell immediately that 7 is not the *absolute* minimum, because the value of the function at the saddle point is lower than 7.

2.1. **Quadratic functions in two variables.** A quadratic function in two variables has the form

$$q(x,y) = ax^2 + bxy + cy^2 + dx + ey + f. \tag{1}$$

To find the critical point(s) for this function we solve the pair of *linear* equations:

$$\begin{cases} q_x &= & 2ax + by + d &= & 0 \\ q_y &= & bx + 2cy + e &= & 0 \end{cases}$$

In general a pair of linear equations in two variables can have either 0, 1 or infinitely many solutions, and this can be determined by the coefficients in the equation. For the pair of equations above, *there is exactly one solutions if and only if*

$$4ac - b^2 \neq 0.$$

Assuming that this is the case, we continue by computing the discriminant in order to apply the second derivative test.

$$q_{xx} = 2a, \quad q_{yy} = 2c, \quad q_{xy} = b \quad \text{and} \quad D = 4ac - b^2.$$

This means that there is exactly one critical point for a quadratic function exactly when its discriminant is not 0!

The critical point gives a saddle point if $D < 0$. If on the other hand, $D > 0$, then the second derivative test implies that the critical point yields a relative maximum (if $a < 0$) or a relative minimum (if $a > 0$). Now, because there is only one critical point that relative maximum or minimum value is actually the **absolute** maximum or minimum!

Summarizing all of this we have the following:

**Fact 3.** *The quadratic function $q(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$ has exactly one critical point $(x_0, y_0)$ if and only if the discriminant $D = 4ac - b^2$ is not equal to 0. In this case, if $D < 0$ ($4ac < b^2$) then the point $(x_0, y_0, q(x_0, y_0))$ is a saddle point on the graph of $z = q(x,y)$. If $D > 0$ ($4ac > b^2$) and $a < 0$ then $q(x_0, y_0)$ is the absolute maximum value of the function, and if $a > 0$ then $q(x_0, y_0)$ is the absolute minimum value of the function.*

**Examples:** $g(x,y) = 2x^2 - 3xy + y^2 + 3x - y + 11$, we have $D = 8 - 9 = -1 < 0$, so we already know that the critical point we will find is a saddle point. (The critical point is $(3,5)$ — verify this!)
$h(u,v) = -3u^2 + 2uv - v^2 + 2u + 6v + 13$, we find the critical point by solving the equations

$$-6u + 2v = -2 \quad \text{and} \quad 2u - 2v = -6.$$

The solution is $(2,5)$. Next we compute $D = 12 - 4 = 8 > 0$, and $h_{uu} = -3 < 0$, so $h(2,5) = 30$ is the absolute maximum value of $h(u,v)$.

## 3. Optimization with constraints

In the previous section we wanted to find extreme values for a given function without any further conditions. This is useful, but in many cases the problem we are trying to solve imposes **constraints** on the variables in question. For example we might want to maximize a production function $Q(x,y,z)$, but we are constrained by a budget that restricts the quantities of the inputs $x$, $y$ and $z$ that we can use. This kind of constraint typically has the form $ax + by + cz = B$.

In the general case (using three variables to be concrete) we want to optimize the function $f(x,y,z)$ subject to the condition

$$g(x,y,z) = c. \tag{2}$$

This means that we only consider triples $(x,y,z)$ that satisfy the equation (2). The constraint imposed by (2) defines a **surface** in three dimensions (three dimensions in this case, because there are three variables). For example, if the constraint is $g(x,y,z) = x^2 + y^2 + z^2 = 9$, then the surface in question is the sphere (surface of a ball) of radius 3, centered at $(0,0,0)$. If $g(x,y,z) = 2x-3y+z$, then the surface in question is a plane. So, in effect, we are looking for the 'best' (smallest or largest) values of $f(x,y,z)$ *on that surface.*

As in the previous section, the first step is to find critical points. If we simply proceed as in §1, by solving the system

$$f_x(x,y,z) = 0, \quad f_y(x,y,z) = 0 \quad \text{and} \quad f_z(x,y,z) = 0,$$

we may find critical points, but there is no guarantee that the points we find will satisfy equation (2). We are looking for a **different kind of critical point** in this scenario, and to find it we use the method of **Lagrange multipliers**[1].

Lagrange's idea was to form the *auxiliary function*, (sometimes called the Lagrangian function),

$$F(x,y,z,\lambda) = f(x,y,z) - \lambda[g(x,y,z) - c]. \tag{3}$$

The new variable $\lambda$ is an auxiliary variable that is called the *multiplier*. Lagrange proved that

**Fact 4.** *The (relative) extreme values of the function $f(x,y,z)$ subject to the constraint $g(x,y,z) = c$ occur at the $(x,y,z)$-coordinates of the critical points of the function*

$$F(x,y,z,\lambda) = f(x,y,z) - \lambda[g(x,y,z) - c].$$

**Comments: (i)** This is a method of finding *relative* extreme values, and other ideas need to be used to further conclude that the values you find are (or are not) *absolute* extreme values.
**(ii)** As far as the basic problem of locating extreme values, the particular value of the auxiliary variable $\lambda$ is usually not important, we only use it to help find $x, y$ and $z$. In applications, however, the multiplier often has important interpretations and uses.

The critical point(s) of $F(x,y,z,\lambda)$ are found the usual way: compute the first order partial derivatives of $F$, set them all equal to 0 and find the solution(s) of the system of equations that this gives. The partial derivatives of $F(x,y,z,\lambda)$ are

$$\begin{aligned}
F_x(x,y,z,\lambda) &= f_x(x,y,z) - \lambda \cdot g_x(x,y,z) \\
F_y(x,y,z,\lambda) &= f_y(x,y,z) - \lambda \cdot g_y(x,y,z) \\
F_z(x,y,z,\lambda) &= f_z(x,y,z) - \lambda \cdot g_z(x,y,z) \\
F_\lambda(x,y,z,\lambda) &= -[g(x,y,z) - c].
\end{aligned}$$

Next, we have to set all four partial derivatives equal to 0. If you do this then you will see that Fact 4 is equivalent to

**Fact 5.** *The extreme values of $f(x,y,z)$ subject to the constraint $g(x,y,z) = c$ occur at the solution(s) of the system of equations*

$$\left.\begin{aligned}
f_x(x,y,z) &= \lambda \cdot g_x(x,y,z), \\
f_y(x,y,z) &= \lambda \cdot g_y(x,y,z), \\
f_z(x,y,z) &= \lambda \cdot g_z(x,y,z), \\
g(x,y,z) &= c.
\end{aligned}\right\} \tag{4}$$

Note that we now have 4 variables and 4 equations. Solutions of this system will be quadruples

$$(x_0, y_0, z_0, \lambda_0),\ (x_1, y_1, z_1, \lambda_1),\ \ldots$$

The critical points that we seek are the $(x,y,z)$-coordinates of these points, namely the triples,

$$(x_0, y_0, z_0),\ (x_1, y_1, z_1),\ \ldots$$

[1] Joseph-Louis Lagrange was a French mathematician, who was active in the second half of the 18th century and the beginning of the 19th century.

**Example 3.1.**

Find critical points for $f(x,y,z) = x^2+y^2+4z^2$ subject to the constraint $x+2y+z = 42$. In this example the constraining function is $g(x,y,z) = x+2y+z$, so we have $f_x = 2x$, $f_y = 2y$, $f_z = 8z$, $g_x = 1$, $g_y = 2$ and $g_z = 1$. Fact 5 gives us the four equations

$$\begin{aligned} 2x &= \lambda \\ 2y &= 2\lambda \\ 8z &= \lambda \\ x+2y+z &= 42 \end{aligned}$$

The first three equations may all be solved simply for $\lambda$ giving

$$\lambda = 2x = y = 8z.$$

From this it follows that $y = 8z$ and $x = 4z$, and we can substitute these expressions in the fourth equation to get

$$4z + 2\cdot 8z + z = 42 \implies 21z = 42 \implies z = 2.$$

So $x = 8$ and $y = 16$ and the critical point in this case is $(8, 16, 2)$.

As in the case of unconstrained optimization problems there do exist *second derivative tests* for constrained optimization problems. Also as in the unconstrained case, these tests become more complicated as the number of variables (and constraints) grows, so for now we will not bother with testing the critical values that we find using Lagrange's method. We will assume that the critical point(s) that we find provide the optimal value(s) that we are seeking.

## 4. Linear constraints

In many applications the constraint is **linear**. This means that equation (2) has the form

$$ax + by + cz = d. \tag{5}$$

In this case the surface defined by the constraint is a plane. The partial derivatives of $g$ are easy to compute: $g_x = a$, $g_y = b$ and $g_z = c$, and inserting these values into the system (4), we obtain the system

$$\left.\begin{aligned} f_x(x,y,z) &= a\lambda, \\ f_y(x,y,z) &= b\lambda, \\ f_z(x,y,z) &= c\lambda, \\ ax+by+cz &= d. \end{aligned}\right\}$$

Assuming that $a, b$ and $c$ are all nonzero[2], we can eliminate $\lambda$ from the first three equations, and obtain the triple equation

$$\frac{f_x}{a} = \frac{f_y}{b} = \frac{f_z}{c}, \tag{6}$$

which, together with the linear constraint, (5), yields the critical points in this scenario. Example 3.1 was like this.

**Example 4.1.**

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[2]If one of them is 0, e.g., $a = 0$, then we still obtain a simple system: $f_x = 0$, $\dfrac{f_y}{b} = \dfrac{f_z}{c}$, and $by + cz = d$. Similar systems arise when one or more of the other constants are 0.

Let $f(x, y, z) = x^2 + 3y^2 + 2z^2$, and find the critical points for this function subject to the constraint

$$x - 2y + 4z = 31.$$

We compute the partial derivatives of $f$ and plug them into the system (6) to obtain

$$\frac{2x}{1} = \frac{6y}{-2} = \frac{4z}{4}.$$

So, $z = 2x$ and $y = -2x/3$. We substitute these values into the constraint, yielding

$$x - 2(-2x/3) + 4(2x) = 31 \implies x = 3,$$

hence $y = -2$, $z = 6$ and the critical point is $(3, -2, 6)$. In this case $f(3, -2, 6) = 93$ is the **minimum** value of $f$ on the plane defined by the constraint. You can verify this by substituting

$$x = 2y + 4z + 31$$

in the function $f(x, y, z)$. This gives the quadratic function of two variables

$$q(y, z) = (2y + 4z + 31)^2 + 3y^2 + 2z^2,$$

(without any constraint!) to which you can apply the methods of §1.

The method of Lagrange multipliers works equally well with functions of 2 variables, 4 variables etc. Let $h(u, v) = u^{3/4} \cdot v^{6/5}$, and let's find the maximum value of this function subject to the constraint $u + v = 13$, and the additional condition that $u > 0$ and $v > 0$. The system (6) reduces to

$$\frac{3}{4}u^{-1/4}v^{6/5} = \frac{6}{5}u^{3/4}v^{1/5}$$

in this case. Since, $u > 0$ and $v > 0$ we may multiply this equation by $u^{1/4}v^{-1/5}$, which gives $3v/4 = 6u/5$, or $v = 24u/15$. Substituting this into the constraint gives $u = 5$ and so $v = 8$. The maximum value is then $h(5, 8) = 40.544\ldots$ .

How do we know that this is the **maximum** value? Well, the constraint together with the additional condition of positivity define the segment in the plane shown below (with the critical point marked for your viewing pleasure).

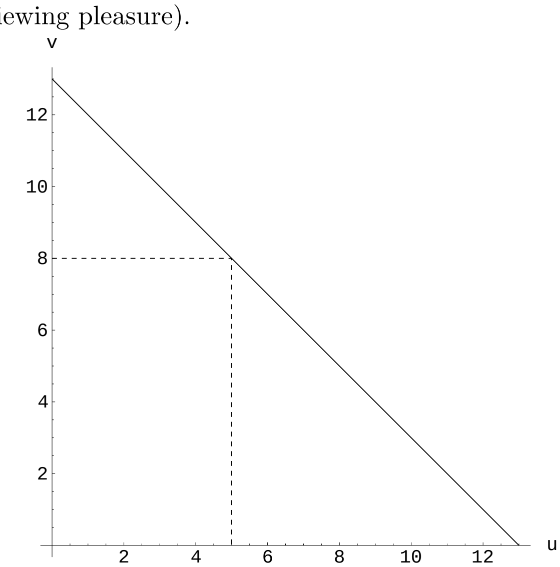


FIGURE 1. Constrained segment for $(u, v)$.

The same basic theorem that guarantees absolute minima and maxima for a continuous function on closed subintervals of the $x$-axis, (section 14.2 in the text), also applies to this case. It is easy to see that the minimum value, 0, occurs at both endpoints, $(0, 13)$ and $(13, 0)$, which means that the maximal value will occur at the critical point that we found.

## 5. Applications

**The Cobb-Douglas** production function. This a function of the form

$$Q = C \cdot u^{\alpha} v^{\beta} w^{\gamma},$$

where $u, v$ and $w$ are the number of units of the inputs, U, V and W used in production. $C$, $\alpha$, $\beta$ and $\gamma$ are strictly positive constants. We assume that each unit of U costs \$$p_u$, each unit of V costs \$$p_v$, each unit of W costs \$$p_w$ and the manufacturer has a total annual budget of \$$B$ to spend on these inputs. The problem is then to decide how to allocate the budget in order to maximize output. This is an archetypical constrained optimization problem: maximize $Q$, subject to the **budget constraint**

$$p_u \cdot u + p_v \cdot v + p_w \cdot w = B.$$

Notice that this budget constraint is linear. We find the partial derivatives of $Q$ and plug them into the triple equation (6) to obtain

$$\frac{\alpha u^{\alpha-1} v^{\beta} w^{\gamma}}{p_u} = \frac{\beta u^{\alpha} v^{\beta-1} w^{\gamma}}{p_v} = \frac{\gamma u^{\alpha} v^{\beta} w^{\gamma-1}}{p_w}. \tag{7}$$

To simplify, we multiply the triple equation by $p_u p_v p_w$ (to clear denominators), and divide by $u^{\alpha-1} v^{\beta-1} w^{\gamma-1}$ (to get rid of the exponents). This leaves us with the kinder, gentler equation

$$\alpha p_v p_w v w = \beta p_u p_w u w = \gamma p_u p_v u v.$$

If we cancel the $w$ from the first two parts of the equation and cancel the $v$ from the first and third parts of the equation and rearrange the constants, we get

$$v = \frac{\beta p_u}{\alpha p_v} \cdot u \quad \text{and} \quad w = \frac{\gamma p_u}{\alpha p_w} \cdot u.$$

Substituting these expressions into the budget constraint above, and solving for $u$, then going back and solving for $v$ and for $w$, we find that production is maximized when[3]

$$u = \frac{\alpha}{\alpha+\beta+\gamma} \cdot \frac{B}{p_u}, \quad v = \frac{\beta}{\alpha+\beta+\gamma} \cdot \frac{B}{p_v}, \quad w = \frac{\gamma}{\alpha+\beta+\gamma} \cdot \frac{B}{p_w}.$$

**Comments:** (a) The partial derivatives $Q_u$, $Q_v$ and $Q_w$ are the *marginal products* of inputs U, V and W, respectively. These are the amounts by which output will increase if one additional unit of the corresponding input is used. The quotients $Q_u/p_u$, $Q_v/p_v$ and $Q_w/p_w$ are the **marginal products of a dollar's worth** of U, V and W respectively. These are the amounts by which output will increase if one additional dollars worth of the corresponding input is used.

Equation (7) says that output is maximized when an extra dollar's worth of U increases output by the same amount as an extra dollar's worth of V or an extra dollar's worth of W. The value of the multiplier $\lambda$ at the critical point is equal to this common value of $Q_u/p_u$, $Q_v/p_v$ and $Q_w/p_w$ when output is maximized. That is $\lambda$ is the *marginal product of \$1*, or more generally, the marginal product of money. Assuming that output is maximized subject to the budget constraint, increasing that budget by \$1 will increase output by $\lambda$ units.

[3] You are strongly encouraged to take the time to work out the simple algebraic steps described here and verify the solution.

Conversely, it will cost $\$1/\lambda$ to produce one additional unit of output, when production is optimized. In other words, $1/\lambda$ is the *marginal cost* for this production function.

(b) The Cobb-Douglas model is not restricted to three variables. It can have as many variables as there are resources that contribute to production. If there are two variables then the function will have the form $Q = Cu^\alpha v^\beta$. If there are $n$ variables, then the function will have the form $Q = Cu_1^{\alpha_1} u_2^{\alpha_2} \cdots u_n^{\alpha_n}$, where $u_k$ is the number of units of the resource $U_k$ being used in the production process. The budget constraint in the case of $n$ resources is

$$p_1 u_1 + p_2 u_2 + \cdots + p_n u_n = B,$$

where $p_k$ is the cost of one unit of $U_k$. Analogously to the case of three variables that we just did, we can apply the method of Lagrange multipliers to show that production is maximized when the number of units of $U_k$ used is

$$u_k = \left( \frac{\alpha_k}{\alpha_1 + \alpha_2 + \cdots + \alpha_n} \right) \cdot \frac{B}{p_k}$$

for each $k$ from 1 to $n$.

**Utility**. Utility functions are used in a variety of situations. A good way to think of 'utility' is as a measure of satisfaction or benefit. For example a household consumes a variety of products, from food, shelter and education to toys, movies and earmuffs. Each product yields a certain benefit and/or satisfaction to the household, and the total utility to the household is a function of the amounts of each product that they consume. Needless to say, most households have finite incomes and so they cannot consume without limit, and a typical problem in this context is to maximize the utility function subject to the income (or budget) constraint.

**Example 5.1.**

Suppose that a utility function is given by

$$U(x, y, z) = 2 \ln x + 3 \ln y + 5 \ln z,$$

where $x, y$ and $z$ are the number of units of commodities X, Y and Z consumed by a household in one month. Find the levels of consumption of these commodities that *maximize utility* if the household's annual budget for these commodities is $B = \$52500$ and the price per unit of X, Y and Z are $p_x = 15$, $p_y = 25$ and $p_z = 35$, respectively.

Simply put, we want to maximize $U(x, y, z)$ subject to the budget constraint

$$p_x \cdot x + p_y \cdot y + p_z \cdot z = 52500.$$

This is a linear constraint and we can use the method outlined in §4. Equation (6) in that section gives the double equation

$$\frac{U_x}{p_x} = \frac{U_y}{p_y} = \frac{U_z}{p_z} \quad (= \lambda), \tag{8}$$

or more explicitly

$$\frac{2}{15x} = \frac{3}{25y} = \frac{5}{35z},$$

because $U_x = \frac{2}{x}$, $U_y = \frac{3}{y}$ and $U_z = \frac{5}{z}$ in this case. By inverting we can solve for $y$ and $z$ in terms of $x$:

$$\frac{25y}{3} = \frac{15x}{2} \implies y = \frac{45}{50}x \quad \text{and} \quad \frac{35z}{5} = \frac{15x}{2} \implies z = \frac{15}{14}x.$$

Substituting these expressions in the budget constraint we obtain

$$15x + 25 \cdot \frac{45}{50}x + 35 \cdot \frac{15}{14}x = 52500 \implies 75x = 52500,$$

so the utility maximizing levels of consumption are $x_0 = 700$, $y_0 = 630$ and $z_0 = 750$.

**Comment.** The expressions $U_x/p_x$, $U_y/p_y$ and $U_z/p_z$ are the **marginal utilities of a dollar's worth** of X, Y and Z respectively. In other words, $U_x/p_x$ is the additional utility that you will gain by spending one more dollar on commodity X, and likewise for Y and Z. Thus, utility is maximized when an extra dollar's worth of X increases utility by the same amount as an extra dollar's worth of Y, or an extra dollar's worth of Z. The *multiplier*, $\lambda$, is the common value of these marginal utilities, and is sometimes called the *marginal utility of income*, (because household budgets are often simply equal to the household disposable income).