

Guía Regla de la Cadena(1<sup>er</sup> Orden)

1. Sean  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  y  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  dos funciones diferenciables.

Se define  $z(u, v) = e^{g(u,v)f^2(u,v)}$ .

- a) Determine las derivadas parciales de  $z$ . Es  $z$  diferenciable?.
- b) Si  $f(u, v) = \sqrt{uv}$  y  $g(u, v) = \frac{1}{v}$ . Compruebe la fórmula obtenida en la parte anterior.

Solución

$$a) \quad \frac{\partial z(u,v)}{\partial u} = e^{g(u,v)f^2(u,v)} \left( \frac{\partial g(u,v)}{\partial u} f^2(u,v) + 2g(u,v)f(u,v) \frac{\partial f(u,v)}{\partial u} \right)$$

$$\frac{\partial z(u,v)}{\partial v} = e^{g(u,v)f^2(u,v)} \left( \frac{\partial g(u,v)}{\partial v} f^2(u,v) + 2g(u,v)f(u,v) \frac{\partial f(u,v)}{\partial v} \right)$$

Notemos que  $z(u, v)$  es composición de funciones diferenciables, y por lo tanto será diferenciable.

- b) Notemos que  $f^2(u, v) = uv$  y  $g(u, v) = \frac{1}{v}$ . Por lo tanto:

$$\frac{\partial f(u,v)}{\partial v} = \frac{u}{2\sqrt{uv}}, \quad \frac{\partial f(u,v)}{\partial u} = \frac{v}{2\sqrt{uv}}$$

$$\frac{\partial g(u,v)}{\partial u} = 0, \quad \frac{\partial g(u,v)}{\partial v} = -\frac{1}{v^2}$$

Con esto  $z(u, v) = e^u$ , con  $\frac{\partial z(u,v)}{\partial u} = e^u$  y  $\frac{\partial z(u,v)}{\partial v} = 0$ . (esto sin utilizar la fórmula)

Con la fórmula:

$$\frac{\partial z(u,v)}{\partial u} = e^u \left( 0 \cdot uv + 2\frac{1}{v} \cdot \sqrt{uv} \cdot \frac{v}{2\sqrt{uv}} \right) = e^u$$

$$\frac{\partial z(u,v)}{\partial v} = e^u \left( -\frac{1}{v^2} \cdot uv + 2\frac{1}{v} \cdot \sqrt{uv} \cdot \frac{u}{2\sqrt{uv}} \right) = 0$$

Con esto comprobamos la fórmula.  $\square$

2. Sean  $f$  y  $g$  dos funciones reales de variable real, derivables en  $\mathbb{R}$ . Se define la función:

$$z(x, y) = x^2 y f\left(\frac{x}{y}\right) + x y^2 g\left(\frac{y}{x}\right)$$

Encuentre el valor de  $E$  dado por la siguiente ecuación:

$$E = x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y}$$

Solución

$$\frac{\partial z(x,y)}{\partial x} = 2xyf\left(\frac{x}{y}\right) + x^2 y f'\left(\frac{x}{y}\right) \frac{1}{y} + y^2 g\left(\frac{y}{x}\right) + x y^2 g'\left(\frac{y}{x}\right) \frac{-y}{x^2}$$

$$\frac{\partial z(x,y)}{\partial y} = x^2 f\left(\frac{x}{y}\right) + x^2 y f'\left(\frac{x}{y}\right) \frac{-x}{y^2} + 2xyg\left(\frac{y}{x}\right) + x y^2 g'\left(\frac{y}{x}\right) \frac{1}{x}$$

Por lo tanto,

$$E = x \cdot (2xyf(\frac{x}{y}) + x^2yf'(\frac{x}{y})\frac{1}{y} + y^2g(\frac{y}{x}) + xy^2g'(\frac{y}{x})\frac{-y}{x^2}) + y \cdot (x^2f(\frac{x}{y}) + x^2yf'(\frac{x}{y})\frac{-x}{y^2} + 2xyg(\frac{y}{x}) + xy^2g'(\frac{y}{x})\frac{1}{x})$$

$$E = 2x^2yf(\frac{x}{y}) + x^3f'(\frac{x}{y}) + xy^2g(\frac{y}{x}) - y^3g'(\frac{y}{x}) + x^2yf(\frac{x}{y}) - x^3f'(\frac{x}{y}) + 2xy^2g(\frac{y}{x}) + y^3g'(\frac{y}{x})$$

$$E = 3x^2yf(\frac{x}{y}) + 3xy^2g(\frac{y}{x})$$

$$E = 3z(x, y)$$

Por lo tanto,  $E = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z(x, y)$ .  $\square$

3. Sean  $g : \mathbb{R} \rightarrow \mathbb{R}$  y  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  funciones diferenciables. Se define la función:

$$f(x, y) = x^2g(\frac{x}{y}) + xyh(\frac{x}{x+y}, \frac{x^2}{y^2}).$$

Demuestre que:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f$$

Solución

$$\frac{\partial f}{\partial x} = 2xg(\frac{x}{y}) + \frac{x^2}{y}g'(\frac{x}{y}) + yh(\frac{x}{x+y}, \frac{x^2}{y^2}) + \frac{xy^2}{(x+y)^2}\frac{\partial h}{\partial x}(\frac{x}{x+y}, \frac{x^2}{y^2}) + 2\frac{x^2}{y}\frac{\partial h}{\partial y}(\frac{x}{x+y}, \frac{x^2}{y^2})$$

$$\frac{\partial f}{\partial y} = \frac{-x^3}{y^2}g'(\frac{x}{y}) + xh(\frac{x}{x+y}, \frac{x^2}{y^2}) - \frac{x^2y}{(x+y)^2}\frac{\partial h}{\partial x}(\frac{x}{x+y}, \frac{x^2}{y^2}) - 2\frac{x^3}{y^2}\frac{\partial h}{\partial y}(\frac{x}{x+y}, \frac{x^2}{y^2})$$

Evaluando la expresión, se obtiene el resultado pedido.  $\square$

4. Sea  $f(\rho, \theta)$  una función de  $\mathbb{R}^2$  en  $\mathbb{R}$  diferenciable. Se define  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  de la siguiente manera:

$$F(x, y) = f(\rho, \theta) = f(\sqrt{x^2 + y^2}, \arctan(\frac{y}{x}))$$

con  $x \neq 0$ .

- a) Demuestre que se cumple

$$\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$

Indique donde están evaluadas las derivadas parciales.

- b) Compruebe la fórmula anterior para  $f(\rho, \theta) = \rho\theta$

Solución

$$\begin{aligned} a) \quad \frac{\partial F}{\partial x} &= \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}, \quad \frac{\partial F}{\partial y} = \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y} \\ \frac{\partial \rho}{\partial x} &= \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}, \quad \frac{\partial \rho}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} \\ \frac{\partial \theta}{\partial x} &= \frac{1}{1+(\frac{y}{x})^2} \cdot -\frac{y}{x^2} = \frac{-y}{x^2+y^2}, \quad \frac{\partial \theta}{\partial y} = \frac{1}{1+(\frac{y}{x})^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2} \end{aligned}$$

Luego,

$$\left(\frac{\partial F}{\partial x}\right)^2 = \left(\frac{\partial f}{\partial \rho} \cdot \frac{x}{\sqrt{x^2+y^2}} + \frac{\partial f}{\partial \theta} \cdot \frac{-y}{x^2+y^2}\right)^2 = \left(\frac{\partial f}{\partial \rho}\right)^2 \cdot \frac{x^2}{x^2+y^2} + \left(\frac{\partial f}{\partial \theta}\right)^2 \cdot \frac{y^2}{(x^2+y^2)^2} - 2\frac{\partial f}{\partial \rho}\frac{\partial f}{\partial \theta} \frac{x}{\sqrt{x^2+y^2}} \frac{-y}{x^2+y^2}$$

$$\left(\frac{\partial F}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial \rho} \cdot \frac{y}{\sqrt{x^2+y^2}} + \frac{\partial f}{\partial \theta} \cdot \frac{x}{x^2+y^2}\right)^2 = \left(\frac{\partial f}{\partial \rho}\right)^2 \cdot \frac{y^2}{x^2+y^2} + \left(\frac{\partial f}{\partial \theta}\right)^2 \cdot \frac{x^2}{(x^2+y^2)^2} + 2 \frac{\partial f}{\partial \rho} \frac{\partial f}{\partial \theta} \frac{y}{\sqrt{x^2+y^2}} \frac{x}{x^2+y^2}$$

Sumando ambos términos:

$$\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial \rho}\right)^2 \cdot \frac{x^2+y^2}{x^2+y^2} + \left(\frac{\partial f}{\partial \theta}\right)^2 \cdot \frac{y^2+x^2}{(x^2+y^2)^2}$$

$$\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial \rho}\right)^2 + \left(\frac{\partial f}{\partial \theta}\right)^2 \cdot \frac{1}{x^2+y^2}$$

Pero, notemos que  $\rho^2 = x^2 + y^2$ , por lo tanto obtenemos:

$$\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial \rho}\right)^2 + \frac{1}{\rho^2} \left(\frac{\partial f}{\partial \theta}\right)^2$$

Las derivadas parciales están evaluadas como sigue:

$\frac{\partial F}{\partial x}$  y  $\frac{\partial F}{\partial y}$  en  $(x, y)$ .

$\frac{\partial f}{\partial \rho}$  y  $\frac{\partial f}{\partial \theta}$  en  $(\sqrt{x^2 + y^2}, \arctan(\frac{y}{x}))$

□

b) Propuesto

5. Sean  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  y  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  diferenciables. Se define la función  $g : \mathbb{R} \rightarrow \mathbb{R}$  por:

$$g(x) = f(x, h(x, x))$$

a) Encuentre  $g'(x)$

b) Compruebe la fórmula anterior para  $f(x, y) = x + 2y$  y  $h(x, y) = 2xy$

Solución

$$a) \quad g'(x) = \frac{\partial f}{\partial x}(x, h(x, x)) + \frac{\partial f}{\partial y}(x, h(x, x)) \left( \frac{\partial h}{\partial x}(x, x) + \frac{\partial h}{\partial y}(x, x) \right)$$

$$b) \quad g(x) = x + 4x^2 \implies g'(x) = 1 + 8x.$$

Evaluando la fórmula se tiene  $g'(x) = 1 + 2(2x + 2x) = 1 + 8x$

□

6. Sean  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  y  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  dos funciones diferenciables.

Si  $h(x, y) = f(x, y, h(x, y))$ , encuentre  $\frac{\partial h}{\partial x}$  y  $\frac{\partial h}{\partial y}$ .

Solución

$$\begin{aligned} \frac{\partial h}{\partial x} &= \frac{\partial f}{\partial x}(x, y, h(x, y)) + \frac{\partial f}{\partial z}(x, y, h(x, y)) \frac{\partial h}{\partial x} \\ \implies \frac{\partial h}{\partial x} &= \frac{\frac{\partial f}{\partial x}(x, y, h(x, y))}{1 - \frac{\partial f}{\partial z}(x, y, h(x, y))} \end{aligned}$$

$$\begin{aligned} \frac{\partial h}{\partial y} &= \frac{\partial f}{\partial y}(x, y, h(x, y)) + \frac{\partial f}{\partial z}(x, y, h(x, y)) \frac{\partial h}{\partial y} \\ \implies \frac{\partial h}{\partial y} &= \frac{\frac{\partial f}{\partial y}(x, y, h(x, y))}{1 - \frac{\partial f}{\partial z}(x, y, h(x, y))} \end{aligned}$$

□

7. Sean  $f : \mathbb{R} \rightarrow \mathbb{R}$  y  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  dos funciones diferenciables tales que  $g(x, y) = xyf(\frac{x+y}{xy})$ . Demuestre que

$$x^2 \frac{\partial g}{\partial x} - y^2 \frac{\partial g}{\partial y} = G(x, y)g(x, y)$$

Encuentre explícitamente a  $G(x, y)$ .

Solución

$$\begin{aligned} \frac{\partial g(x, y)}{\partial x} &= yf\left(\frac{x+y}{xy}\right) + xyf'\left(\frac{x+y}{xy}\right) \cdot \frac{xy-y(x+y)}{x^2y^2} = yf\left(\frac{x+y}{xy}\right) - f'\left(\frac{x+y}{xy}\right) \cdot \frac{y}{x} \\ \frac{\partial g(x, y)}{\partial y} &= xf\left(\frac{x+y}{xy}\right) + xyf'\left(\frac{x+y}{xy}\right) \cdot \frac{xy-x(x+y)}{x^2y^2} = xf\left(\frac{x+y}{xy}\right) - f'\left(\frac{x+y}{xy}\right) \cdot \frac{x}{y} \end{aligned}$$

Luego,

$$\begin{aligned} x^2 \frac{\partial g}{\partial x} - y^2 \frac{\partial g}{\partial y} &= x^2 yf\left(\frac{x+y}{xy}\right) - xyf'\left(\frac{x+y}{xy}\right) - xy^2 f\left(\frac{x+y}{xy}\right) + xyf'\left(\frac{x+y}{xy}\right) \\ x^2 \frac{\partial g}{\partial x} - y^2 \frac{\partial g}{\partial y} &= x^2 yf\left(\frac{x+y}{xy}\right) - xy^2 f\left(\frac{x+y}{xy}\right) = (x - y)xyf\left(\frac{x+y}{xy}\right) = G(x, y)g(x, y) \end{aligned}$$

De donde podemos identificar  $G(x, y) = x - y$ .

□

8. Sea  $h : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  una función definida por:

$$z(x, y) = xy \tan\left(\frac{y}{x}\right)$$

Determine, para  $(x, y) \neq (0, 0)$  (en términos de  $z$ ) el valor de la siguiente expresión:

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y}$$

Explique, además, que sucede en  $(0, 0)$ .

Solución

$$\begin{aligned} \frac{\partial z}{\partial x} &= y \tan\left(\frac{y}{x}\right) - \frac{y^2}{x} \sec^2\left(\frac{y}{x}\right) \\ \frac{\partial z}{\partial y} &= x \tan\left(\frac{y}{x}\right) + y \sec^2\left(\frac{y}{x}\right) \end{aligned}$$

Evalutando la expresión se tiene:

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2xy \tan\left(\frac{y}{x}\right) = 2z(x, y)$$

En el origen, la función  $z(x, y)$  no está definida, por lo tanto no tiene derivadas parciales.

□

9. Se define la función  $f(x, y, z) = \left(\frac{x-y+z}{x+y-z}\right)^n$ .

Determine el valor de

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z}$$

si  $(x, y, z) \neq (0, 0, 0)$

Solución

$$\frac{\partial f(x, y, z)}{\partial x} = n \left(\frac{x-y+z}{x+y-z}\right)^{n-1} \cdot \frac{x+y-z-(x-y+z)}{(x+y-z)^2} = n \left(\frac{x-y+z}{x+y-z}\right)^{n-1} \cdot \frac{2y-2z}{(x+y-z)^2}$$

$$\begin{aligned}\frac{\partial f(x,y,z)}{\partial y} &= n \left( \frac{x-y+z}{x+y-z} \right)^{n-1} \cdot \frac{-(x+y-z)-(x-y+z)}{(x+y-z)^2} = n \left( \frac{x-y+z}{x+y-z} \right)^{n-1} \cdot \frac{-2x}{(x+y-z)^2} \\ \frac{\partial f(x,y,z)}{\partial z} &= n \left( \frac{x-y+z}{x+y-z} \right)^{n-1} \cdot \frac{x+y-z+(x-y+z)}{(x+y-z)^2} = n \left( \frac{x-y+z}{x+y-z} \right)^{n-1} \cdot \frac{2x}{(x+y-z)^2}\end{aligned}$$

Luego, obtenemos:

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = n \left( \frac{x-y+z}{x+y-z} \right)^{n-1} \cdot \frac{2xy-2xz}{(x+y-z)^2} + n \left( \frac{x-y+z}{x+y-z} \right)^{n-1} \cdot \frac{-2xy}{(x+y-z)^2} + n \left( \frac{x-y+z}{x+y-z} \right)^{n-1} \cdot \frac{2xz}{(x+y-z)^2}$$

Por lo tanto,

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = 0$$

□

10. Se define la función  $G : \mathbb{R}^3 \rightarrow \mathbb{R}$  por  $G(x, y, z) = z \sin\left(\frac{y}{x}\right)$ . Si hacemos el cambio de variable  $x = 3r^2 + 2s$ ,  $y = 4r - 2s^3$  y  $z = 2r^2 - 3s^2$ , entonces  $G$  se transforma en una función  $\tilde{G} : \mathbb{R}^2 \rightarrow \mathbb{R}$ .

Encuentre  $\frac{\partial \tilde{G}}{\partial r}$  y  $\frac{\partial \tilde{G}}{\partial s}$  evaluadas en  $r$  y  $s$ , pero en función de  $x, y, z$ .

Solución

Lo que se indica en el enunciado, es que mediante el cambio de variable se forma una función  $\tilde{G}(r, s) = G(3r^2 + 2s, 4r - 2s^3, 2r^2 - 3s^2)$ .

Luego,

$$\begin{aligned}\frac{\partial \tilde{G}}{\partial r} &= \frac{\partial G}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial G}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial G}{\partial z} \frac{\partial z}{\partial r} \\ \frac{\partial \tilde{G}}{\partial s} &= \frac{\partial G}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial G}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial G}{\partial z} \frac{\partial z}{\partial s}.\end{aligned}$$

Recordemos que las derivadas parciales están evaluadas de la siguiente forma:

$$\begin{aligned}\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \text{ y } \frac{\partial G}{\partial z} &\text{ en } (3r^2 + 2s, 4r - 2s^3, 2r^2 - 3s^2). \\ \frac{\partial x}{\partial r}, \frac{\partial y}{\partial r} \text{ y } \frac{\partial z}{\partial r} &\text{ en } (r, s).\end{aligned}$$

Calculemos las derivadas parciales que necesitamos:

$$\begin{aligned}\frac{\partial G}{\partial x} &= z \cos\left(\frac{y}{x}\right) \cdot \frac{-y}{x^2}, \quad \frac{\partial G}{\partial y} = z \cos\left(\frac{y}{x}\right) \cdot \frac{1}{x}, \quad \frac{\partial G}{\partial z} = \sin\left(\frac{y}{x}\right) \\ \frac{\partial x}{\partial r} &= 6r, \quad \frac{\partial y}{\partial r} = 4, \quad \frac{\partial z}{\partial r} = 4r \\ \frac{\partial x}{\partial s} &= 2, \quad \frac{\partial y}{\partial s} = -6s^2, \quad \frac{\partial z}{\partial s} = -6s\end{aligned}$$

Con lo recién calculado, estamos listos con el problema. Ahora simplemente debemos reemplazar. La continuación queda propuesta.

□

11. Pruebe que la función  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  definida por  $F(u, v) = f(uv, \frac{u^2-v^2}{2})$  donde  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  es una función diferenciable, verifica la siguiente igualdad:

$$\left(\frac{\partial F}{\partial u}\right)^2 + \left(\frac{\partial F}{\partial v}\right)^2 = (u^2 + v^2) \left( \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \right)$$

Indique donde están evaluadas las derivadas parciales.

Solución

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}$$

Notemos, además que:

$$\begin{aligned} \frac{\partial x}{\partial v} &= u, \quad \frac{\partial x}{\partial u} = v \\ \frac{\partial y}{\partial v} &= -v, \quad \frac{\partial y}{\partial u} = u \end{aligned}$$

Luego,

$$\begin{aligned} \left(\frac{\partial F}{\partial u}\right)^2 &= \left(v \frac{\partial f}{\partial x} + u \frac{\partial f}{\partial y}\right)^2 = v^2 \left(\frac{\partial f}{\partial x}\right)^2 + u^2 \left(\frac{\partial f}{\partial y}\right)^2 + 2uv \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \\ \left(\frac{\partial F}{\partial v}\right)^2 &= \left(u \frac{\partial f}{\partial x} - v \frac{\partial f}{\partial y}\right)^2 = u^2 \left(\frac{\partial f}{\partial x}\right)^2 + v^2 \left(\frac{\partial f}{\partial y}\right)^2 - 2uv \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \end{aligned}$$

Finalmente, sumando ambos términos obtenemos que:

$$\left(\frac{\partial F}{\partial u}\right)^2 + \left(\frac{\partial F}{\partial v}\right)^2 = (u^2 + v^2) \left( \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \right)$$

□

12. Sea  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  definida por:

$$F(u, v, w) = \begin{pmatrix} f_1(u, v, w) \\ f_2(u, v, w) \\ f_3(u, v, w) \end{pmatrix}$$

donde  $f_i : \mathbb{R}^3 \rightarrow \mathbb{R}$   $i = 1, 2, 3$ .

Se define la *divergencia* de  $F$  como:

$$\operatorname{div} F = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$$

Por otro lado, sea  $G : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  definida por:

$$G(r, \theta, z) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \\ z \end{pmatrix}$$

Se definen además los vectores unitarios  $\hat{r}$ ,  $\hat{\theta}$  y  $\hat{z}$  como sigue:

$$\hat{r} = \frac{\frac{\partial G}{\partial r}}{\left\| \frac{\partial G}{\partial r} \right\|}, \quad \hat{\theta} = \frac{\frac{\partial G}{\partial \theta}}{\left\| \frac{\partial G}{\partial \theta} \right\|}, \quad \hat{z} = \frac{\frac{\partial G}{\partial z}}{\left\| \frac{\partial G}{\partial z} \right\|}$$

donde  $\frac{\partial G}{\partial r}$ ,  $\frac{\partial G}{\partial \theta}$  y  $\frac{\partial G}{\partial z}$  son vectores que se obtienen derivando parcialmente a  $G$  componente a componente.

Considere, por último,  $H : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  definida como:

$$H(r, \theta, z) = h_1(r, \theta, z)\hat{r} + h_2(r, \theta, z)\hat{\theta} + h_3(r, \theta, z)\hat{z}$$

a) Encuentre los vectores unitarios  $\hat{r}$ ,  $\hat{\theta}$  y  $\hat{z}$ .

- b) Deduzca que  $H$  se puede escribir como:

$$H(r, \theta, z) = A_1(r, \theta, z)\hat{i} + A_2(r, \theta, z)\hat{j} + A_3(r, \theta, z)\hat{z}$$

donde  $\hat{i}$ ,  $\hat{j}$  y  $\hat{z}$  representan los vectores unitarios canónicos.  
Encuentre explícitamente  $A_1$ ,  $A_2$  y  $A_3$ .

- c) Demuestre que

$$\text{div} H = \frac{1}{r} \left( \frac{\partial(rh_1)}{\partial r} + \frac{\partial h_2}{\partial \theta} \right) + \frac{\partial h_3}{\partial z}$$

Solución

$$\begin{aligned} a) \quad \frac{\partial G}{\partial r} &= \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix} \text{ y } \left\| \frac{\partial G}{\partial r} \right\| = 1 \Rightarrow \hat{r} = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix} \\ \frac{\partial G}{\partial \theta} &= \begin{pmatrix} -r \sin \theta \\ r \cos \theta \\ 0 \end{pmatrix} \text{ y } \left\| \frac{\partial G}{\partial \theta} \right\| = r \Rightarrow \hat{\theta} = \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix} \\ \frac{\partial G}{\partial z} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ y } \left\| \frac{\partial G}{\partial z} \right\| = 1 \Rightarrow \hat{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

- b) Utilizando la parte anterior, tenemos que  $H$  se puede escribir como:

$$H(r, \theta, z) = (h_1 \cos \theta - h_2 \sin \theta)\hat{i} + (h_1 \sin \theta + h_2 \cos \theta)\hat{j} + h_3\hat{z}$$

De donde,

$$A_1 = h_1 \cos \theta - h_2 \sin \theta$$

$$A_2 = h_1 \sin \theta + h_2 \cos \theta$$

$$A_3 = h_3$$

- c) Notemos que de la definición de "divergencia" tenemos que:

$$\text{div} H = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}.$$

Notemos además que  $r = \sqrt{x^2 + y^2}$  y que  $\theta = \arctan\left(\frac{y}{x}\right)$ , entonces:

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{\sqrt{x^2 + y^2}} = \frac{r \cos \theta}{r} = \cos \theta, \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \frac{r \sin \theta}{r} = \sin \theta \\ \frac{\partial \theta}{\partial x} &= \frac{-y}{x^2 + y^2} = \frac{-r \sin \theta}{r^2} = \frac{-\sin \theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2} = \frac{r \cos \theta}{r^2} = \frac{\cos \theta}{r} \end{aligned}$$

Calculemos las derivadas:

$$\begin{aligned} \frac{\partial A_1}{\partial x} &= \left( \frac{\partial h_1}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial h_1}{\partial \theta} \frac{\partial \theta}{\partial x} \right) \cos \theta - h_1 \sin \theta \frac{\partial \theta}{\partial x} - \left( \frac{\partial h_2}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial h_2}{\partial \theta} \frac{\partial \theta}{\partial x} \right) \sin \theta - h_2 \cos \theta \frac{\partial \theta}{\partial x} \\ \frac{\partial A_2}{\partial y} &= \left( \frac{\partial h_1}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial h_1}{\partial \theta} \frac{\partial \theta}{\partial y} \right) \sin \theta + h_1 \cos \theta \frac{\partial \theta}{\partial y} + \left( \frac{\partial h_2}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial h_2}{\partial \theta} \frac{\partial \theta}{\partial y} \right) \cos \theta - h_2 \sin \theta \frac{\partial \theta}{\partial y} \\ \frac{\partial A_3}{\partial z} &= \frac{\partial h_3}{\partial z} \end{aligned}$$

Reemplacemos ahora, las derivadas parciales:

$$\begin{aligned}\frac{\partial A_1}{\partial x} &= \left( \frac{\partial h_1}{\partial r} \cos \theta + \frac{\partial h_1}{\partial \theta} \frac{-\sin \theta}{r} \right) \cos \theta - h_1 \sin \theta \frac{-\sin \theta}{r} - \left( \frac{\partial h_2}{\partial r} \cos \theta + \frac{\partial h_2}{\partial \theta} \frac{-\sin \theta}{r} \right) \sin \theta - h_2 \cos \theta \frac{-\sin \theta}{r} \\ \frac{\partial A_2}{\partial y} &= \left( \frac{\partial h_1}{\partial r} \sin \theta + \frac{\partial h_1}{\partial \theta} \frac{\cos \theta}{r} \right) \sin \theta + h_1 \cos \theta \frac{\cos \theta}{r} + \left( \frac{\partial h_2}{\partial r} \sin \theta + \frac{\partial h_2}{\partial \theta} \frac{\cos \theta}{r} \right) \cos \theta - h_2 \sin \theta \frac{\cos \theta}{r} \\ \frac{\partial A_3}{\partial z} &= \frac{\partial h_3}{\partial z}\end{aligned}$$

Finalmente sumando las 3 ecuaciones anteriores se obtiene:

$$\operatorname{div} H = \frac{\partial h_1}{\partial r} + \frac{h_1}{r} + \frac{1}{r} \frac{\partial h_2}{\partial \theta} + \frac{\partial h_3}{\partial z}$$

Lo cual nos conduce a:

$$\operatorname{div} H = \frac{1}{r} \left( \frac{\partial(rh_1)}{\partial r} + \frac{\partial h_2}{\partial \theta} \right) + \frac{\partial h_3}{\partial z}$$

□

13. Sean  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  tales que  $f''$  y  $g''$  existen. Sea  $F(x, y) = f(x + g(y))$  Calcular  $\frac{\partial^2 F}{\partial x \partial y}$  y  $\frac{\partial^2 F}{\partial y^2}$  ¿Dónde están evaluadas?.

Solución

$$\begin{aligned}\frac{\partial F}{\partial y} &= f'(x + g(y))g'(y) \\ \frac{\partial^2 F}{\partial y^2} &= f''(x + g(y))[g'(y)]^2 + f'(x + g(y))g''(y) \\ \frac{\partial^2 F}{\partial x \partial y} &= f''(x + g(y))g'(y) \quad \square\end{aligned}$$

14. Sea  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$  de clase  $C^2(\mathbb{R}^2)$ . Demuestre que la ecuación

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$$

toma la forma

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial s^2} = 0$$

bajo el cambio de variables  $x = e^r$   $y = e^s$

Solución

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} \\ \frac{\partial}{\partial r} \left( \frac{\partial u}{\partial r} \right) &= \frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r} \left( \frac{\partial u}{\partial x} \right) \frac{\partial x}{\partial r} + \frac{\partial u}{\partial x} \frac{\partial^2 x}{\partial r^2} \\ \implies \frac{\partial^2 u}{\partial r^2} &= \frac{\partial^2 u}{\partial x^2} \left( \frac{\partial x}{\partial r} \right)^2 + \frac{\partial u}{\partial x} \frac{\partial^2 x}{\partial r^2} = \frac{\partial^2 u}{\partial x^2} x^2 + \frac{\partial u}{\partial x} x\end{aligned}$$

Por simetría se tiene que:

$$\frac{\partial^2 u}{\partial s^2} = \frac{\partial^2 u}{\partial y^2} y^2 + \frac{\partial u}{\partial y} y$$

Sumando ambas expresiones se obtiene el resultado pedido.

□



15. Sea  $z : \mathbb{R}^2 \rightarrow \mathbb{R}$  de clase  $C^2(\mathbb{R}^2)$ . Demuestre que la ecuación

$$2\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$$

se transforma en

$$3\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial z}{\partial u} = 0$$

bajo el cambio de variables  $u = x + 2y + 2$   $v = x - y - 1$

Solución

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial y} \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \\ \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial y} \frac{\partial v}{\partial y} = 2\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial u} \right) + \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial v} \right) = \left( \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u} \right) + \left( \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right) \\ \implies \frac{\partial^2 z}{\partial x^2} &= \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x \partial y} &= 2\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial u} \right) - \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial v} \right) = 2\left( \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u} \right) - \left( \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right) \\ \implies \frac{\partial^2 z}{\partial x \partial y} &= 2\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial y^2} &= 2\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial u} \right) - \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial v} \right) = 2\left( 2\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v \partial u} \right) - \left( 2\frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v^2} \right) \\ \implies \frac{\partial^2 z}{\partial y^2} &= 4\frac{\partial^2 z}{\partial u^2} - 4\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\end{aligned}$$

Reemplazando en la expresión se obtiene el resultado pedido.

□