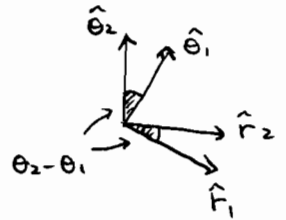


Determine el movimiento del Sistema para pequeñas oscilaciones

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$$\vec{r}_1 = l_1 \hat{r}_1 \rightarrow \dot{\vec{r}}_1 = l_1 \dot{\theta}_1 \hat{\theta}_1$$

$$\vec{r}_2 = l_1 \hat{r}_1 + l_2 \hat{r}_2 \rightarrow \dot{\vec{r}}_2 = l_1 \dot{\theta}_1 \hat{\theta}_1 + l_2 \dot{\theta}_2 \hat{\theta}_2$$



$$T = \frac{1}{2} m_1 \dot{\vec{r}}_1^2 + \frac{1}{2} m_2 \dot{\vec{r}}_2^2$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 [l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \underbrace{\hat{\theta}_1 \cdot \hat{\theta}_2}_{\cos(\theta_1 - \theta_2)}]$$

$$= \frac{1}{2} \underbrace{(m_1 + m_2)}_{\equiv M} l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$V = -m_1 g l_1 \cos \theta_1 - m_2 g (l_1 \cos \theta_1 + l_2 \cos \theta_2)$$

$$= -\underbrace{(m_1 + m_2)}_{\equiv M} g l_1 \cos \theta_1 - m_2 g l_2 \cos \theta_2$$

• Ptos de equilibrio : $\frac{\partial V}{\partial \theta_i} = 0 \quad i=1,2$

$$\frac{\partial V}{\partial \theta_1} = M g l_1 \sin \theta_1 = 0 \Rightarrow \theta_1 = 0, \pi$$

$$\frac{\partial V}{\partial \theta_2} = m_2 g l_2 \sin \theta_2 = 0 \Rightarrow \theta_2^* = 0, \pi$$

• Estabilidad :

Si $\begin{bmatrix} \frac{\partial^2 V}{\partial \theta_1^2} & \frac{\partial^2 V}{\partial \theta_1 \partial \theta_2} \\ \frac{\partial^2 V}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 V}{\partial \theta_2^2} \end{bmatrix}_{(\theta_1^*, \theta_2^*)}$ es definida positiva $\Rightarrow (\theta_1^*, \theta_2^*)$ es pto de eq. estable

$$\frac{\partial^2 V}{\partial \theta_1^2} = M g l_1 \cos \theta_1$$

$$\frac{\partial^2 V}{\partial \theta_2 \partial \theta_1} = \frac{\partial^2 V}{\partial \theta_1 \partial \theta_2} = 0$$

$$\frac{\partial^2 V}{\partial \theta_2^2} = m_2 g l_2 \cos \theta_2$$

$$\Rightarrow \begin{bmatrix} \frac{\partial^2 V}{\partial \theta_1^2} & \frac{\partial^2 V}{\partial \theta_2 \partial \theta_1} \\ \frac{\partial^2 V}{\partial \theta_1 \partial \theta_2} & \frac{\partial^2 V}{\partial \theta_2^2} \end{bmatrix} = \begin{bmatrix} M g l_1 \cos \theta_1 & 0 \\ 0 & m_2 g l_2 \cos \theta_2 \end{bmatrix} \quad (*)$$

$$\Rightarrow (\theta_1^*, \theta_2^*) = \begin{cases} (0,0) & \text{eq. estable} \\ (0,\pi) & \text{pto silla} \\ (\pi,0) & \text{pto silla} \\ (\pi,\pi) & \text{eq. inestable} \end{cases}$$

- Aproximación pequeñas oscilaciones

$$\text{En general, } T = \frac{1}{2} \sum_{i,j} m_{ij} (q_1, \dots, q_n) \dot{q}_i \dot{q}_j$$

$$\approx \frac{1}{2} \sum_{i,j} m_{ij} \Big|_{\substack{\text{pto} \\ \text{eq}}} \dot{q}_i \dot{q}_j$$

$$V \approx V \Big|_{\substack{\text{pto} \\ \text{eq}}} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 V}{\partial q_i \partial q_j} \Big|_{\substack{\text{pto} \\ \text{eq}}} q_i q_j$$

En nuestro caso, el pto de eq es $(\theta_1, \theta_2) = (0, 0)$

$$T \approx \frac{1}{2} M l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \Big|_{(\theta_1, \theta_2) = (0, 0)} \dot{\theta}_1 \dot{\theta}_2$$

$$= \frac{1}{2} M l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2$$

$$= \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \underbrace{\begin{bmatrix} M l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{bmatrix}}_{\equiv \tilde{T}} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$y \quad V \approx \underbrace{V|_{(\theta_1, \theta_2) = (0,0)}}_{\text{cte que no interesa}} + \frac{1}{2} [\theta_1 \ \theta_2] \underbrace{\left[\frac{\partial^2 V}{\partial \theta_i \partial \theta_j} \right]}_{(\theta_1, \theta_2) = (0,0)} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

$$\text{de (*)} \Rightarrow V = \text{cte} + \frac{1}{2} [\theta_1 \ \theta_2] \underbrace{\begin{bmatrix} M g l_1 & 0 \\ 0 & m_2 g l_2 \end{bmatrix}}_{\equiv \tilde{V}} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

$$\text{Entonces, } L = T - V$$

$$= \frac{1}{2} \dot{\vec{\theta}}^T \tilde{T} \dot{\vec{\theta}} - \frac{1}{2} \vec{\theta}^T \tilde{V} \vec{\theta} + \text{cte} ; \quad \vec{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \quad \text{con}$$

La ecuación de movimiento correspondiente es:

$$\tilde{T} \ddot{\vec{\theta}} + \tilde{V} \vec{\theta} = 0$$

$$\text{Ansatz: } \vec{\theta}(t) = \vec{A} \cdot \cos(\omega t + \phi)$$

$$\Rightarrow -\tilde{T} \omega^2 \vec{A} \cos(\omega t + \phi) + \tilde{V} \vec{A} \cos(\omega t + \phi) = 0 \quad \forall t$$

$$\Rightarrow (\tilde{V} - \omega^2 \tilde{T}) \cdot \vec{A} = 0 \quad (**)$$

$$\text{Solución no trivial cuando } \det(\tilde{V} - \omega^2 \tilde{T}) = 0$$

$$\Rightarrow \begin{vmatrix} M g l_1 - M l_1^2 \omega^2 & -m_2 l_1 l_2 \omega^2 \\ -m_2 l_1 l_2 \omega^2 & m_2 g l_2 - m_2 l_2^2 \omega^2 \end{vmatrix} = 0$$

$$\Rightarrow (M g l_1 - M l_1^2 \omega^2)(m_2 g l_2 - m_2 l_2^2 \omega^2) - m_2^2 l_1^2 l_2^2 \omega^4 = 0 \quad \left/ \cdot \frac{1}{m_2 l_1 l_2} \right.$$

$$\Rightarrow (M g - M l_1 \omega^2)(g - l_2 \omega^2) - m_2 l_1 l_2 \omega^4 = 0$$

$$\Rightarrow M g^2 - (M g l_2 + M g l_1) \omega^2 + M l_1 l_2 \omega^4 - m_2 l_1 l_2 \omega^4 = 0$$

$$\Rightarrow \underbrace{(M - m_2)}_{m_1} l_1 l_2 \omega^4 - M g (l_1 + l_2) \omega^2 + M g^2 = 0$$

$$\Rightarrow \omega^2 = \frac{Mg(l_1 + l_2) \pm \sqrt{M^2 g^2 (l_1 + l_2)^2 - 4m_1 l_1 l_2 M g^2}}{2m_1 l_1 l_2}$$

$$\Rightarrow \omega^2 \begin{cases} \omega_+^2 \\ \omega_-^2 \end{cases}$$

$$\text{Sea } \vec{A} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

entonces $(**)$ queda

$$\begin{bmatrix} Mgl_1 - Ml_1^2 \omega^2 & -m_2 l_1 l_2 \omega^2 \\ -m_2 l_1 l_2 \omega^2 & m_2 gl_2 - m_2 l_2^2 \omega^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0$$

Las dos ecuaciones anteriores son l.d. Por comodidad, tomamos la segunda ecuación

$$-m_2 l_1 l_2 \omega^2 \alpha + (m_2 gl_2 - m_2 l_2^2 \omega^2) \beta = 0 \quad / \cdot 1/m_2 l_2$$

$$-l_1 \omega^2 \alpha + (g - l_2 \omega^2) \beta = 0$$

$$\Rightarrow \alpha = \frac{g - l_2 \omega^2}{l_1 \omega^2} \beta$$

$$\text{Para } \omega_+ : \alpha = \frac{g - l_2 \omega_+^2}{l_1 \omega_+^2} \beta \Rightarrow \vec{A}_+ = \begin{pmatrix} \frac{g - l_2 \omega_+^2}{l_1 \omega_+^2} \\ 1 \end{pmatrix}$$

$$\text{Para } \omega_- : \alpha = \frac{g - l_2 \omega_-^2}{l_1 \omega_-^2} \beta \Rightarrow \vec{A}_- = \begin{pmatrix} \frac{g - l_2 \omega_-^2}{l_1 \omega_-^2} \\ 1 \end{pmatrix}$$

$$\therefore \vec{\theta}(t) = \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \end{bmatrix} = C_1 \begin{bmatrix} \frac{g - l_2 \omega_+^2}{l_1 \omega_+^2} \\ 1 \end{bmatrix} \cos(\omega_+ t + \phi_1) + C_2 \begin{bmatrix} \frac{g - l_2 \omega_-^2}{l_1 \omega_-^2} \\ 1 \end{bmatrix} \cos(\omega_- t + \phi_2)$$