Pauta Ej 2 FI21B 2005/02 Chen

P] $L = \underbrace{\frac{1}{2} I_1 (\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \frac{1}{2} I_3 (\dot\psi + \dot\phi \cos\theta)^2}_{T} - \underbrace{Mgl\cos\theta}_{V}$

(a) En $t=0$, $\dot\psi = \dot\psi_0$, $\dot\phi = \dot\theta = \theta = 0$

- $\frac{\partial L}{\partial \phi} = 0 \Rightarrow \frac{\partial L}{\partial \dot\phi} = \text{cte}$

$$\Rightarrow \boxed{I_1 \dot\phi \sin^2\theta + I_3(\dot\psi + \dot\phi\cos\theta)\cdot\cos\theta \overset{t=0}{=} I_3 \dot\psi_0} \quad (1)$$

- $\frac{\partial L}{\partial \psi} = 0 \Rightarrow \frac{\partial L}{\partial \dot\psi} = \text{cte}$ 0,5

$$\Rightarrow \boxed{I_3(\dot\psi + \dot\phi\cos\theta) \overset{t=0}{=} I_3 \dot\psi_0} \quad (2)$$

- $\frac{\partial L}{\partial t} = 0 \Rightarrow H = \text{cte}$

$\frac{\partial \vec r(\theta,\phi,\psi;t)}{\partial t} = 0 \Rightarrow H = E = T + V$

$$\Rightarrow \boxed{\begin{aligned} E &= \frac{1}{2} I_1 (\dot\theta^2 + \dot\phi^2\sin^2\theta) + \frac{1}{2} I_3 (\dot\psi + \dot\phi\cos\theta)^2 + Mgl\cos\theta \\ &\overset{t=0}{=} \frac{1}{2} I_3 \dot\psi_0^2 + Mgl \end{aligned}} \quad (3)$$

0,5

(b) Hay que demostrar que si $\dot\psi_0^2 > 4mgl I_1/I_3^2$

$\Rightarrow \theta = 0$ es un pto de equilibrio estable

(1) $\rightarrow$ $I_1 \dot\phi \sin^2\theta + \underbrace{I_3(\dot\psi + \dot\phi\cos\theta)}_{I_3\dot\psi_0}\cdot\cos\theta = I_3\dot\psi_0$

$$\Rightarrow \dot\phi = \frac{I_3\dot\psi_0 (1-\cos\theta)}{I_1 \sin^2\theta} \qquad 0,5$$

(3) $\rightarrow$ $E = \frac{1}{2} I_1 (\dot\theta^2 + \dot\phi^2\sin^2\theta) + \frac{1}{2} I_3 (\underbrace{\dot\psi + \dot\phi\cos\theta}_{\dot\psi_0})^2 + Mgl\cos\theta$

$\dot\phi$ en E $\Rightarrow$ $E = \frac{1}{2} I_1 \dot\theta^2 + \frac{1}{2} \frac{I_3^2\dot\psi_0^2}{I_1} \Big(\underbrace{\frac{1-\cos\theta}{\sin\theta}}_{\frac{\sin\theta}{1+\cos\theta}}\Big)^2 + \frac{1}{2} I_3 \dot\psi_0 + Mgl\cos\theta$

$$\Rightarrow E' \equiv E - \frac{1}{2} I_3 \dot\psi_0^2$$

$$= \frac{1}{2} I_1 \dot\theta^2 + \underbrace{\frac{1}{2}\frac{I_3^2}{I_1}\dot\psi_0^2\left(\frac{\sin\theta}{1+\cos\theta}\right)^2 + Mgl\cos\theta}_{\equiv V_{ef}(\theta)\quad 0,5} \overset{t=0}{=} Mgl \qquad (4)$$

$$V_{ef}'(\theta) = \frac{I_3^2}{I_1}\dot\psi_0^2\left(\frac{\sin\theta}{1+\cos\theta}\right)\cdot\left[\frac{\cos\theta(1+\cos\theta)+\sin^2\theta}{(1+\cos\theta)^2}\right] - Mgl\sin\theta$$

$$= \frac{I_3^2}{I_1}\dot\psi_0^2\,\frac{\sin\theta}{(1+\cos\theta)^2} - Mgl\sin\theta$$

$$\Rightarrow \boxed{V_{ef}'(0) = 0}$$

$\Rightarrow \theta = 0$ es pto crítico 0,5

$$V_{ef}''(\theta) = \frac{I_3^2}{I_1}\dot\psi_0^2\left[\frac{\cos\theta(1+\cos\theta)^2 - \sin\theta\cdot 2(1+\cos\theta)(-\sin\theta)}{(1+\cos\theta)^4}\right] - Mgl\cos\theta \quad (5)$$

$$\Rightarrow V_{ef}''(0) = \frac{I_3^2}{I_1}\dot\psi_0^2\,\frac{1}{4} - Mgl$$

Si $\dot\psi_0^2 > 4Mgl\,I_1/I_3^2 \Rightarrow \boxed{V_{ef}''(\theta=0) > 0}$

$\Rightarrow \theta = 0$ es mínimo 0,5

(c) Hay que determinar los puntos de retorno

$$\dot\theta = 0 \text{ en } (4) \Rightarrow \frac{1}{2}\frac{I_3^2}{I_1}\dot\psi_0^2\left(\frac{\sin\theta}{1+\cos\theta}\right)^2 = Mgl(1-\cos\theta) \quad /\cdot(1+\cos\theta)^2$$

$$\Rightarrow \frac{1}{2}\frac{I_3^2}{I_1}\dot\psi_0^2\sin^2\theta - Mgl\underbrace{(1-\cos\theta)(1+\cos\theta)}_{\sin^2\theta}(1+\cos\theta) = 0$$

$$\Rightarrow \left[\frac{1}{2}\frac{I_3^2}{I_1}\dot\psi_0^2 - Mgl(1+\cos\theta)\right]\cdot\sin^2\theta = 0$$

$$\Rightarrow \boxed{\theta_1 = 0 \quad \text{ó} \quad \theta_2 = \cos^{-1}\left(\frac{\dot\psi_0^2 I_3^2}{2MglI_1} - 1\right)} \qquad 2,0$$

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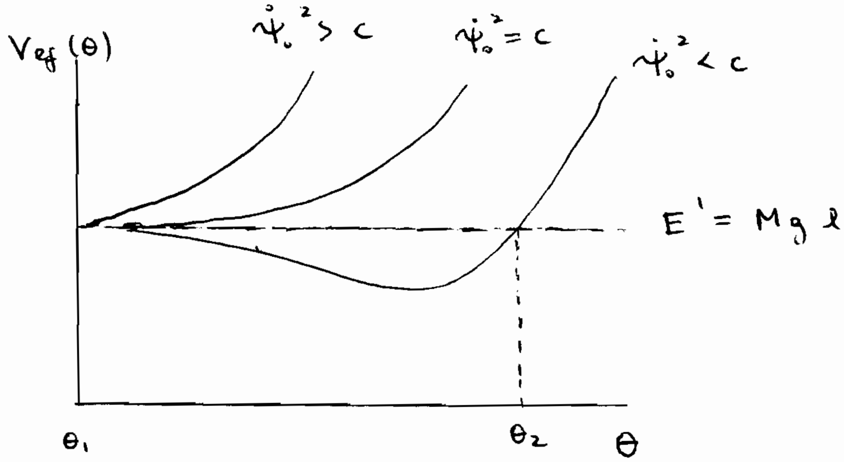
nota: Una forma alternativa de hacer la parte (b) es obteniendo la ecuación de movimiento de θ mediante E-L e identificando $V_{ef}'(\theta)$.

(d)



$C = 4 M g \ell I_1 / I_3^2$

1,0