24/8/2005

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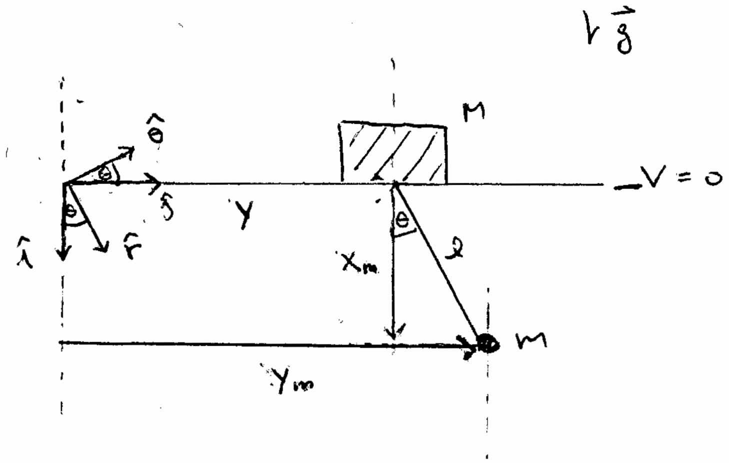

Determinar las ecuaciones de movimiento y alguna expresión para la tensión en la cuerda.

$$\vec{r}_M = y\,\hat{j} \longrightarrow \dot{\vec{r}}_M = \dot{y}\,\hat{j}$$

$$\vec{r}_m = y\,\hat{j} + r\,\hat{r} \longrightarrow \dot{\vec{r}}_m = \dot{y}\,\hat{j} + \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}$$

$$T = \frac{1}{2}M\dot{\vec{r}}_M^{\,2} + \frac{1}{2}m\,\dot{\vec{r}}_m^{\,2}$$

$$= \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m\Big(\dot{y}^2\underbrace{\hat{j}\cdot\hat{j}}_{1} + \dot{r}^2\underbrace{\hat{r}\cdot\hat{r}}_{1} + r^2\dot{\theta}^2\underbrace{\hat{\theta}\cdot\hat{\theta}}_{1} + 2\dot{y}\dot{r}\underbrace{\hat{j}\cdot\hat{r}}_{\sin\theta} + 2\dot{y}r\dot{\theta}\underbrace{\hat{j}\cdot\hat{\theta}}_{\cos\theta} + 2\dot{r}r\dot{\theta}\underbrace{\hat{r}\cdot\hat{\theta}}_{0}\Big)$$

$$= \frac{1}{2}(M+m)\dot{y}^2 + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + m\dot{y}\dot{r}\sin\theta + m\dot{y}r\dot{\theta}\cos\theta$$

$$V = -mg\,x_m = -mgr\cos\theta$$

Constricción: $r = \ell \longrightarrow C(y, r, \theta) = r - \ell = 0$

Entonces,

$$L = \frac{1}{2}(M+m)\dot{y}^2 + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + m\dot{y}\dot{r}\sin\theta + m\dot{y}r\dot{\theta}\cos\theta +$$

$$+ mgr\cos\theta + \lambda(r-\ell)$$

• Para r

$$\frac{\partial L}{\partial \dot r} = m\dot r + m\dot y \sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = m\ddot r + m\ddot y \sin\theta + m\dot y\dot\theta\cos\theta$$

$$\frac{\partial L}{\partial r} = mr\dot\theta^2 + m\dot y\dot\theta\cos\theta + mg\cos\theta + \lambda$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) - \frac{\partial L}{\partial r} = 0 \Rightarrow m\ddot r + m\ddot y\sin\theta - mr\dot\theta^2 - mg\cos\theta - \lambda = 0$$

$$r = \ell = cte \Rightarrow \boxed{m\ddot y\sin\theta - m\ell\dot\theta^2 - mg\cos\theta = \lambda} \qquad (1)$$

• Para y

Conviene imponer de inmediato que $r=\ell$

$$L = \frac{1}{2}(M+m)\dot y^2 + \frac{1}{2}m\ell^2\dot\theta^2 + m\dot y\ell\dot\theta\cos\theta + mg\ell\cos\theta$$

$$\frac{\partial L}{\partial \dot y} = (M+m)\dot y + m\ell\dot\theta\cos\theta$$

$$\frac{\partial L}{\partial y} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot y}\right) - \frac{\partial L}{\partial y} = 0 \Rightarrow \frac{d}{dt}\Big(\underbrace{(M+m)\dot y + m\ell\dot\theta\cos\theta}_{\equiv P_y}\Big) = 0$$

$\equiv P_y$ momentum lineal en el eje y se conserva

$$\Rightarrow \boxed{(M+m)\dot y + m\ell\dot\theta\cos\theta = P_y = cte} \qquad (2)$$

• Para θ

$$\frac{\partial L}{\partial \dot\theta} = m\ell^2\dot\theta + m\dot y\ell\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = m\ell^2\ddot\theta + m\ddot y\ell\cos\theta - m\dot y\ell\dot\theta\sin\theta$$

$$\frac{\partial L}{\partial \theta} = -m\dot y\ell\dot\theta\sin\theta - mg\ell\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \cancel{m} l^2 \ddot\theta + \cancel{m} \ddot y l \cos\theta + \cancel{m} g l \sin\theta = 0$$

$$\Rightarrow \boxed{\ddot\theta + \frac{\ddot y}{l}\cos\theta + \frac{g}{l}\sin\theta = 0} \qquad (3)$$

Las ecuaciones (2) y (3) determinan completamente (salvo condiciones iniciales) el movimiento del sistema.

Para determinar la tensión de la cuerda usamos:

$$\vec f_i = \sum \lambda_k \cdot \nabla_i C_k$$

En nuestro caso:

$$\vec T_m = \lambda \cdot \nabla_{\vec r_m} C = \lambda \left(\frac{\partial C}{\partial X_m}, \frac{\partial C}{\partial Y_m} \right)$$

donde $C = r - l = \sqrt{X_m^2 + (Y_m - Y)^2} - l$

$$\Rightarrow \frac{\partial C}{\partial X_m} = \frac{X_m}{\sqrt{X_m^2 + (Y_m - Y)^2}} = \frac{\cancel{r}\cos\theta}{\cancel{r}} = \cos\theta$$

$$y \quad \frac{\partial C}{\partial Y_m} = \frac{Y_m - Y}{\sqrt{X_m^2 + (Y - Y_m)^2}} = \frac{\cancel{r}\cdot\sin\theta}{\cancel{r}} = \sin\theta$$

$$\Rightarrow \vec T_m = \lambda \underbrace{(\cos\theta, \sin\theta)}_{\hat r(\theta)}$$

$$\Rightarrow \boxed{\vec T_m = (m \ddot y \sin\theta - m l \dot\theta^2 - m g \cos\theta)\cdot \hat r(\theta)}$$