PREGUNTA 1

1.1 $Y = \frac{X}{a} \Rightarrow F_Y(y) = P(Y \le y) = P(X \le ay) = F_X(ay) \Rightarrow$

$f_Y(y) = af_X(ay) = a\beta e^{-a\beta x} \Rightarrow Y \sim \exp(a\beta)$

1.2 $L = \beta^n e^{-\beta \sum x_i} \Rightarrow \log(L) = n\log(\beta) - \beta\sum x_i$

$\frac{\partial \log(L)}{\partial \beta} = 0 \Rightarrow \frac{n}{\hat{\beta}} = \sum x_i \Rightarrow \hat{\beta} = \frac{1}{\bar{x}}$.

1.3 $X_i \sim N(1,4). \Rightarrow \bar{x} \sim N(1,\frac{4}{n})$. **Si** $Z \sim N(0,1). \Rightarrow P(|Z| \le 1.96) = 0.95$

$\Rightarrow P(\bar{x} \in [1 - 1.96\frac{2}{\sqrt{n}}, 1 + 1.96\frac{2}{\sqrt{n}}]) = 0.95$.

1.4 Si se repiten 100 muestras, se esperan que 95 de las muestras tengan su media muestral que cae en el intervalo.

PREGUNTA 2

2.1 $E(\hat{\mu}) = \sum_{i=1}^{n} a_i E(X_i) + \sum_{j=1}^{m} b_j E(Y_j) = \mu(\sum_{i=1}^{n} a_i + \sum_{j=1}^{m} b_j) = \mu \Rightarrow \boxed{\sum_{i=1}^{n} a_i + \sum_{j=1}^{m} b_j = 1}$

2.2 $V(\hat{\mu}) = \sum_{i=1}^{n} a_i^2 Var(X_i) + \sum_{j=1}^{m} b_j^2 Var(Y_j) = \sigma^2(\sum_{i=1}^{n} a_i^2 + 2\sum_{j=1}^{m} b_j^2)$. Hay que minimizar

$Var(\hat{\mu})$ sujeto a $\sum_{i=1}^{n} a_i + \sum_{j=1}^{m} b_j = 1 \Rightarrow Q = Var(\hat{\mu}) - \lambda(\sum_{i=1}^{n} a_i + \sum_{j=1}^{m} b_j - 1)$

$$\left.\begin{array}{l} \frac{\partial Q}{\partial a_i} = 0 \Rightarrow 2\sigma^2 a_i - \lambda = 0 \Rightarrow a_i = \frac{\lambda}{2\sigma^2} \ (a_i \text{ todos iguales}) \\ \frac{\partial Q}{\partial b_j} = 0 \Rightarrow 4\sigma^2 b_j - \lambda = 0 \Rightarrow b_j = \frac{\lambda}{4\sigma^2} \ (b_j \text{ todos iguales}) \end{array}\right\} \Rightarrow a_i = 2b_j$$

De $\sum_{i=1}^{n} a_i + \sum_{j=1}^{m} b_j = 1$ se deduce que $\boxed{a_i = \frac{2}{2n+m},\ b_j = \frac{1}{2n+m}}$ y

$\boxed{\hat{\mu} = \left(\frac{1}{2n+m}\right)(2\sum_i X_i + \sum_j Y_j)}$ Es decir que es una media ponderada de todas las observaciones, en donde los X_i tiene un peso doble del peso de los Y_j.

2.3 Calculemos la cota inferior de la desigualdad de Cramer-Rao. La función de verosimilitud de las (n+m) valores muestrales es:

$$L=\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp(-\frac{1}{2\sigma^2}\sum_i (X_i-\mu)^2)\left(\frac{1}{\sqrt{4\pi\sigma^2}}\right)^m \exp(-\frac{1}{4\sigma^2}\sum_j (Y_j-\mu)^2)$$

$$\log(L)=-\left(\frac{n}{2}\right)\log(2\pi\sigma^2)-\left(\frac{m}{2}\right)\log(4\pi\sigma^2)-\frac{1}{\sigma^2}\{\frac{1}{2}\sum_i (X_i-\mu)^2)+\frac{1}{4}\sum_j (Y_j-\mu)^2\}$$

$$\frac{\partial \log(L)}{\partial\mu}=\frac{1}{\sigma^2}\{\sum_i (X_i-\mu))+\frac{1}{2}\sum_j (Y_j-\mu)\}$$

$$\frac{\partial^2 \log(L)}{\partial\mu^2}=-\frac{1}{\sigma^2}\{n+\frac{m}{2}\}=-\frac{1}{\sigma^2}\{\frac{2n+m}{2}\} \Rightarrow \boxed{Var(\hat{\mu})\geq\frac{2\sigma^2}{2n+m}}$$

Por otro lado:

$Var(\hat{\mu})=\sigma^2(\sum_i a_i^2+2\sum_j b_j^2)=\dfrac{2\sigma^2}{2n+m} \Rightarrow \hat{\mu}$ es un estimador insesgado de mínima varianza para μ.

2.4 Se tiene $\mu^*=\sum_{i=1}^n a_iX_i+\sum_{j=1}^m b_jY_j$ y $E\{(\mu^*-\mu)^2\}=Var(\mu^*)+\{E(\mu^*-\mu))\}^2$.

$Var(\mu^*)=\sigma^2(\sum_i a_i^2+2\sum_j b_j^2)$ y $E(\mu^*-\mu)=\mu(\sum_i a_i+\sum_j b_j-1)$

$$E\{(\mu^*-\mu)^2\}=\sigma^2(\sum_i a_i^2+2\sum_j b_j^2)+\mu^2(\sum_i a_i+\sum_j b_j-1)^2$$

Minimizando:

$$\frac{\partial E\{(\mu^*-\mu)^2\}}{\partial a_i}=2\sigma^2 a_i+2\mu^2(\sum_i a_i+\sum_j b_j-1)=0 \Rightarrow a_i=\frac{\mu^2}{\sigma^2}(1-\sum_i a_i-\sum_j b_j)$$

$$\frac{\partial E\{(\mu^*-\mu)^2\}}{\partial b_j}=2\sigma^2 b_j+2\mu^2(\sum_i a_i+\sum_j b_j-1)=0 \Rightarrow b_j=\frac{\mu^2}{2\sigma^2}(1-\sum_i a_i-\sum_j b_j)$$

Luego todos los b_j son iguales y todos los a_i son iguales a $2b_j$. Sea $b_j=b$ y $a_i=2b$.

Entonces $\sum_i a_i=\dfrac{n\mu^2}{\sigma^2}(1-\sum a_i-\sum b_j)$ o sea $2nb=\dfrac{n\mu^2}{\sigma^2}(1-2nb-mb)$

Finalmente: $b=\dfrac{\mu^2}{2\sigma^2+(2n+m)\mu^2}$ y $\mu^*=\sum_{i=1}^n a_iX_i+\sum_{j=1}^m b_jY_j=2b\sum X_i+b\sum Y_j$

$$\mu^*=b(2\sum X_i+\sum Y_j)=\frac{\mu^2}{2\sigma^2+(2n+m)\mu^2}(2\sum X_i+\sum Y_j)$$

2.5 De aquí se deduce la varianza de μ^*:

$$Var(\mu^*)=\left(\frac{\mu^2}{2\sigma^2+(2n+m)\mu^2}\right)^2(4n\sigma^2+m\sigma^2)=\frac{2(2n+m)\sigma^2}{(2\sigma^2/\mu^2+(2n+m))^2}$$

Como $Var(\hat{\mu})=\dfrac{2\sigma^2}{2n+m}=\dfrac{2(2n+m)\sigma^2}{(2n+m)^2}$, se deduce que $\boxed{Var(\hat{\mu})>Var(\mu^*)}$

PREGUNTA 3

3.1 $\xi(\theta \mid X)=\begin{cases}\dfrac{\theta+2(1-\theta)x}{2g(x)} & \text{si } 0\le\theta\le 2\\ 0 & \text{si no}\end{cases}$ con $g(x)=\int_0^2 \frac{1}{2}(\theta+2(1-\theta)xd\theta=1$

Luego $\xi(\theta \mid X)=\begin{cases}\dfrac{\theta+2(1-\theta)x}{2} & \text{si } 0\le\theta\le 2\\ 0 & \text{si no}\end{cases}$

3.2 Bajo perdida cuadrática, $\theta_B = E(\theta \mid x)=\int_0^2 \frac{\theta}{2}(\theta+2(1-\theta)x)d\theta=\frac{4}{3}-\frac{2}{3}x$

3.3 $E(\theta_B \mid x)=\frac{4}{3}-\frac{2}{3}E(x)$

$$E(x)=\int_0^1 x(\theta+2(1-\theta)x)dx=\theta\frac{x^2}{2}+2(1-\theta)\frac{x^3}{3}\Big|_0^1=(\frac{\theta}{2}+\frac{2(1-\theta)}{3}=\frac{2}{3}-\frac{\theta}{6}$$

Luego $E(\theta_B \mid x)=\frac{4}{3}-\frac{2}{3}E(x)=\frac{4}{3}-\frac{2}{3}\left(\frac{2}{3}-\frac{\theta}{6}\right)=\frac{\theta+8}{9}\Rightarrow$ el estimador es sesgado.

3.4 Si $x=1/2$,

$$P(0\le\theta\le k \mid x=1/2)=\int_0^k \xi(\theta \mid x=1/2)d\theta=\int_0^k \frac{1}{2}(\theta(1-\theta)d\theta=\int_0^k \frac{1}{2}d\theta=\frac{k}{2}$$

$P(0\le\theta\le k \mid x=1/2)=1/2 \;\Rightarrow \boxed{k=1}$