FIN 653 Portfolio Analysis and Management Professor Kishinevsky
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LECTURE NOTES

Determining Efficient Frontier

- Efficient Frontier With No Short Sales
- Efficient Frontier With Riskless Lending and Borrowing
- Analytical Solution of the Portfolio Selection Problem
- Efficient Frontier With Additional Constraints

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Efficient Frontier with No Short Sales

- We can trace efficient frontier with no short sales using techniques similar to those used when short sales were allowed. We can start with finding the global minimum variance portfolio:

$$\text{Minimize: } \sigma_P^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \sigma_{ij} = \mathbf{x}\mathbf{S}\mathbf{x}^{\mathbf{T}}$$

Subject to:

1. $$\sum_{i=1}^{N} x_i = 1$$
2. $x_i \geq 0 \quad i = 1, \ldots, N$

- Notice, we have new constraint: all portfolio weights must be positive.
- Similarly we can find the *maximum return portfolio*:

$$\text{Maximize: } \overline{R}_P = \sum_{i=1}^{N} x_i \overline{R}_i = \mathbf{x}\overline{\mathbf{R}}^{\mathbf{T}}$$

Subject to:

1. $$\sum_{i=1}^{N} x_i = 1$$
2. $x_i \geq 0 \quad i = 1, \ldots, N$

Efficient Frontier with No Short Sales

- When short sales are not permitted all the efficient portfolios will lie between the minimum variance portfolio and the maximum return portfolio. Now we can trace the efficient frontier by solving a series of optimization problems for a series of target portfolio returns R_{pj}:

$$\text{Minimize:} \quad \sigma_P^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \sigma_{ij} = \mathbf{x}^{\mathbf{T}}\mathbf{S}\,\mathbf{x}$$

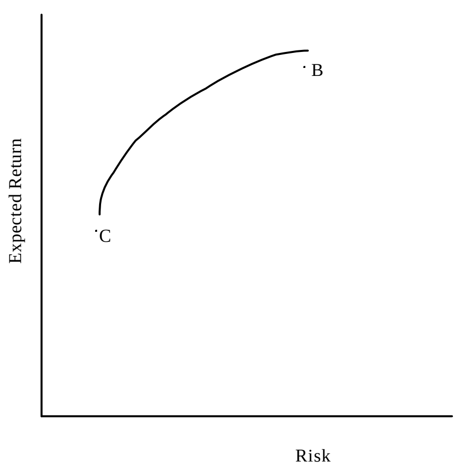
Subject to:

$$1. \quad \sum_{i=1}^{N} x_i = 1$$

$$2. \quad x_i \geq 0 \quad i = 1, \ldots, N$$

$$3. \quad \overline{R}_P = \sum_{i=1}^{N} x_i \overline{R}_i = \mathbf{x}\overline{\mathbf{R}}^{\mathbf{T}} = R_{Pj}$$

Efficient Frontier with No Short Sales



- Portfolio C - global minimum variance portfolio
- Portfolio B - maximum return portfolio

Markowitz's Corner Portfolios

- Efficient frontier contains infinite number of portfolios.
- In 1952 Markowitz developed a *critical-line method* for determining the efficient frontier.
- Critical-line algorithm identifies a number of *corner portfolios* that completely describe the whole efficient frontier. Every efficient portfolio is a linear combination of two corner portfolios immediately adjacent to it.
- Details of the critical-line algorithm are beyond the scope of this course. Interested students can find description of the algorithm in Macro-Investment Analysis by William Sharpe (http://www.stanford.edu/~wfsharpe/mia/opt/mia_opt3.htm)
- Critical-line method can be implemented in Visual Basic as a custom Excel function. For example, Corner Portfolios spreadsheet (http://webpage.pace.edu/mkishinevsky/software/djia-cp.xls) provides custom function =CRITLINEOPT for determining a set of corner portfolios.

Markowitz's Corner Portfolios

- =CRITLINEOPT function has the following syntax:
  =CRITLINEOPT(lbd,ubd,er,cv,stats), where
  - *lbd* is a lower boundary. It can be a number, single cell, or row or column range.
  - *ubd* is an upper boundary. It can be a number, single cell, or row or column range.
  - *er* is a row or column range of expected returns.
  - *cv* is a variance-covariance matrix.
  - *stats* is a logical value specifying whether to return additional statistics (default value is FALSE).
- Standard "no short sales" problem has lbd=0, and ubd=1.
- =CRITLINEOPT function returns an array of values. If we have N stocks and solution produces M corner portfolios, the array has M+1 rows and either N (if stats is FALSE) or N+1 (if stats is TRUE) columns. The first cell of the first row reports the number of corner portfolios. Rows 2 to M+1 report compositions of corner portfolios. If stats is TRUE, the last column of rows 2 to M+1 contains risk tolerances of corner portfolios.

Efficient Frontier with Riskless Lending and Borrowing

Portfolio return:

$$\overline{R}_P = X\overline{R}_A + (1-X)R_F$$

Portfolio risk:

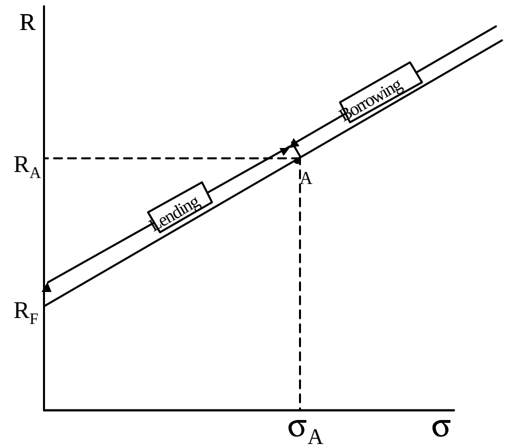
$$\sigma_P^2 = X^2\sigma_A^2 + (1-X)^2\sigma_F^2 - 2X(1-X)\rho_{AF}\sigma_A\sigma_F$$

Since σ_F=0,

$$\sigma_P = X\sigma_A$$

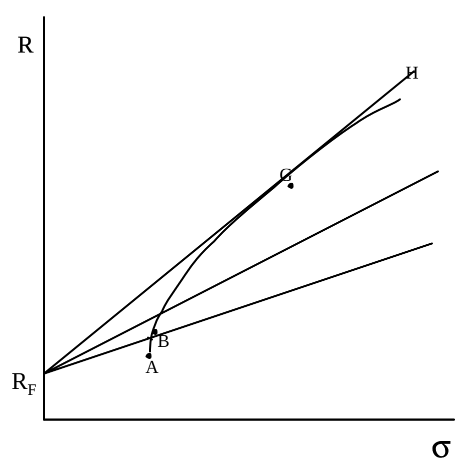
Solving for X, and substituting into the expression for return yields

$$\overline{R}_P = R_F + \frac{(\overline{R}_A - R_F)}{\sigma_A}\sigma_P$$

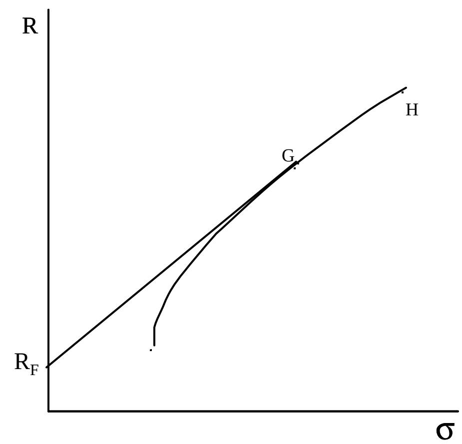


Efficient Frontier with Riskless Lending and Borrowing

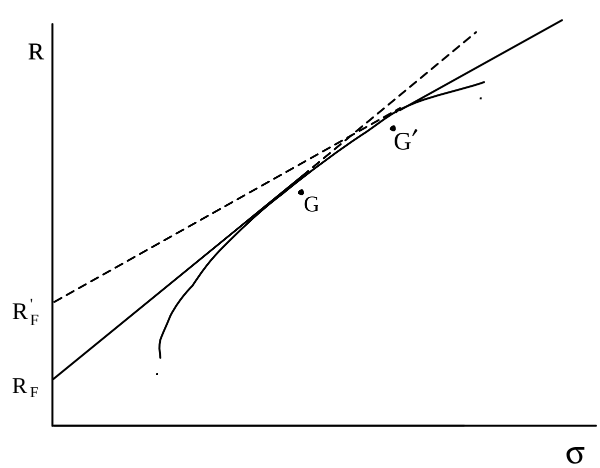
- Portfolio A in the previous example could be any portfolio
- Combination along R_FB are superior to combination along R_FA
- There is no combination superior to R_FG
- Portfolio G - *market portfolio*
- Line R_FGH - *capital market line*
- Separation theorem
  - All investors who believed they faced the efficient frontier and riskless lending and borrowing rates would hold the same portfolio of risky assets.



Efficient Frontier with Lending but Not Borrowing At Riskless Rate



Efficient Frontier with Riskless Lending and Borrowing at Different Rates



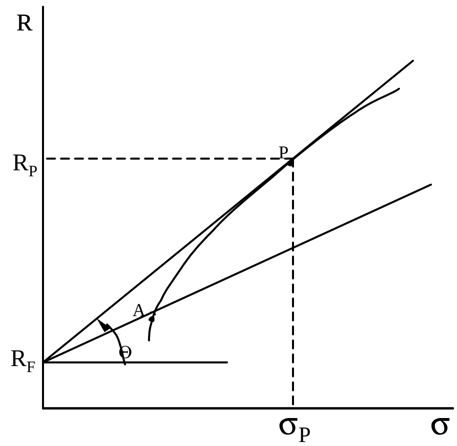
Analytical Solution

- There is a single portfolio of risky assets which is preferred to all other portfolios
- Optimum portfolio P can be found by solving the following problem:

$$\text{Maximize}: \quad \Theta = \frac{\bar{R}_P - R_F}{\sigma_P}$$

$$\text{Subject to}: \quad \sum_{i=1}^{N} X_i = 1$$

- This is constrained optimization problem
- We can solve this problem analytically when
  - Short sales are allowed
  - We can borrow and lend at the risk-free rate



Analytical Solution

We can write R_F as R_F times 1

$$R_F = (1)R_F = \sum_{i=1}^{N} X_i R_F$$

Substitute it into the objective function

$$\Theta = \frac{\sum_{i=1}^{N} X_i \left(\bar{R}_i - R_F\right)}{\left(\sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} X_i X_j \sigma_{ij}\right)^{1/2}}$$

Theta reaches maximum in the point where

$$\frac{d\Theta}{dX_1} = 0$$

$$\frac{d\Theta}{dX_2} = 0$$

$$\vdots$$

$$\frac{d\Theta}{dX_N} = 0$$

Analytical Solution

Application of simple rules of differentiation yields:

$$\frac{d\Theta}{dX_k} = -\lambda\sum_{i=1}^{N} X_i\sigma_{ki} + R_k - R_F = 0$$

Making substitution $Z_k=\lambda X_k$ we get the following system of N equations:

$$R_k - R_F = \sum_{i=1}^{N} Z_i\sigma_{ik}$$

Or, equivalently:

$$\begin{aligned} R_1 - R_F &= Z_1\sigma_1^2 + Z_2\sigma_{12} + \cdots + Z_N\sigma_{1N} \\ R_2 - R_F &= Z_1\sigma_{21} + Z_2\sigma_2^2 + \cdots + Z_N\sigma_{2N} \\ &\vdots \\ R_N - R_F &= Z_1\sigma_{N1} + Z_2\sigma_{N2} + \cdots + Z_N\sigma_N^2 \end{aligned}$$

After solving this system for Z_k, we can find X_k from:

$$X_k = \frac{Z_k}{\sum_{i=1}^{N} Z_i}$$

Analytical Solution: Determining the Derivative

For two-asset portfolio:

$$\Theta = \frac{\overline{R}_P - R_F}{\sigma_P} = \frac{X_1R_1 + X_2R_2 - X_1R_F - X_2R_F}{\left(X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_2\sigma_{12}\right)^{1/2}}$$

Let us write Θ as:

$$\Theta = u \times v \quad u = \overline{R}_P - R_F \quad v = \sigma_P^{-1}$$

According to the derivative multiplication rule:

$$\frac{d\Theta}{dX_1} = \frac{du}{dX_1}\times v + \frac{dv}{dX_1}\times u$$

Derivative of **u** with respect to X_1:

$$\frac{du}{dX_1} = R_1 - R_F$$

Analytical Solution: Determining the Derivative

In order to find derivative of **v** with respect to X_1 we will use the derivative chain rule:

$$\frac{dF(G(x))}{dx} = \frac{dF}{dG} \times \frac{dG}{dx}$$

Application of the chain rule yields:

$$\frac{dv}{dX_1} = (-1)\left(\sigma_P^{-2}\right)\frac{d\sigma_P}{dX_1}$$

Use the chain rule, once again, to find derivative of σ_P with respect to X_1:

$$\frac{d\sigma_P}{dX_1} = \frac{1}{2}\left(X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_2\sigma_{12}\right)^{-\frac{1}{2}}\left(2X_1\sigma_1^2 + 2X_2\sigma_{12}\right) = \frac{X_1\sigma_1^2 + X_2\sigma_{12}}{\sigma_P}$$

Combining these results we get:

$$\frac{d\Theta}{dX_1} = \frac{R_1 - R_F}{\sigma_P} - \frac{\overline{R}_P - R_F}{\sigma_P^3}\left(X_1\sigma_1^2 + X_2\sigma_{12}\right)$$

Analytical Solution: Determining the Derivative

We can write constant term as following:

$$\lambda = \frac{\overline{R}_P - R_F}{\sigma_P^2}$$

Further simplifying we can write $d\Theta/dX_1=0$ as:

$$R_1 - R_F = \lambda X_1\sigma_1^2 + \lambda X_2\sigma_{12}$$

Following the same steps we can simplify $d\Theta/dX_2=0$ to:

$$R_2 - R_F = \lambda X_1\sigma_{12} + \lambda X_2\sigma_2^2$$

In order to find X_1 and X_2, substitute:

$$Z_1 = \lambda X_1 \quad Z_2 = \lambda X_2$$

Solve system for Z_1 and Z_2, and find X_1 and X_2 from:

$$X_1 = \frac{Z_1}{Z_1 + Z_2} \quad X_2 = \frac{Z_2}{Z_1 + Z_2}$$

Analytical Solution: Implementation with Excel

- Let us consider the system of equations for Z-coefficients:

$$
\begin{aligned}
Z_1\sigma_1^2 + Z_2\sigma_{12} + \cdots + Z_N\sigma_{1N} &= R_1 - R_F \\
Z_1\sigma_{21} + Z_2\sigma_2^2 + \cdots + Z_N\sigma_{2N} &= R_2 - R_F \\
&\vdots \\
Z_1\sigma_{N1} + Z_2\sigma_{N2} + \cdots + Z_N\sigma_N^2 &= R_N - R_F
\end{aligned}
$$

- Using matrix notation we can write this equation as:

$$\mathbf{Z} \times \mathbf{S} = \mathbf{R} - R_F$$

Where:

Z – row vector of Z-coefficients

S – variance/covariance matrix

R – row vector of expected returns

- Solution to this system is found by multiplying left and right part of the equation by inverse of variance/covariance matrix:

$$\mathbf{Z} = (\mathbf{R} - R_F) \times \mathbf{S}^{-1}$$

Analytical Solution: Implementation with Excel

- Consider familiar example of four assets:

Asset	AA	EK	JNJ	MSFT
Mean Return	-0.000806341	0.011140271	0.021681	0.056378775
Standard Deviation	0.096023339	0.058905929	0.066868	0.106090009
Variance/Covariance Matrix				
	0.009220482	0.002771406	0.003461	0.001676178
	0.002771406	0.003469908	0.000841	-2.2102E-05
	0.003460818	0.000841261	0.004471	0.004967685
	0.001676178	-2.2102E-05	0.004968	0.01125509

- Assume that risk free rate is 5% per year, or .05/12 = 0.00417 per month. By defining named range *rfr* for the cell with annual risk free rate we can find row vector of Z-coefficients using the following Excel formula: =MMULT((mean-rfr/12),MINVERSE(S))
- Then we can name the row of Z-coefficients as *z_1*, and compute the weights of the tangent portfolio as: =z_1/SUM(z_1)

Risk Free Rate	0.05			
Z-vector	-2.372021388	4.223528619	-1.171890421	5.517767455
X-vector	-0.382745572	0.681501814	-0.189094362	0.89033812
		Risk	Return	
Risk-Free Asset		0	0.05	
Tangent Portfolio, rfr = 0.05		0.310623151	0.647965417	

Analytical Solution: Implementation with Excel

- If riskless lending and borrowing is not available we still can trace efficient frontier without use of the Solver. We can compute composition of two tangent portfolios using arbitrary risk-free rates, and then trace efficient frontier using these two portfolios.
- For example we can select second risk-free rate at 60%:

Alternative r.f.r	0.6			
Z-vector	0.127835666	-8.706967595	-10.56373096	5.193147899
X-vector	-0.009164034	0.624168135	0.757272171	-0.37227627
		Risk	Return	
Risk-Free Asset		0	0.6	
Tangent Portfolio, rfr = 0.6		0.202373209	0.028690595	
Asset	A: rfr=0.05	B: rfr=0.6		
Mean Return	0.647965417	0.028690595		
Std. Dev.	0.310623151	0.202373209		
Covariance	-0.003438451			
		Weight A	Weight B	
0.310623151	0.647965417	1	0	
0.302623002	0.632483546	0.975	0.025	
0.294711756	0.617001676	0.95	0.05	
0.286896767	0.601519805	0.925	0.075	

Incorporation of Additional Constraints

- So far we have considered two types of constraints
  - Constraints on portfolio return
  - Constraints on portfolio risk
- But we can formulate and solve optimization problem that takes into account the following constraints:
  - Portfolio dividend yield
  - Upper limit of portfolio weights
  - Lower limit on portfolio weights
  - Upper limit of investment into particular industry
  - Limit on transaction costs

Incorporation of Additional Constraints

- For example, portfolio manager might be required to select a minimum variance portfolio that has target expected return R_P and target dividend yield D_P. In this case optimization problem can be formulated as following:

Minimize: $\sigma_P = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \sigma_{ij}} = \sqrt{\mathbf{xSx}^T}$

By changing : $\mathbf{x}$

Subject to: $\sum_{i=1}^{N} x_i = 1$

$$\sum_{i=1}^{N} x_i \overline{R}_i = \mathbf{x}\overline{\mathbf{R}}^T = R_P$$

$$\sum_{i=1}^{N} x_i d_i = \mathbf{xd}^T \geq D_P$$

Where :

$\overline{\mathbf{R}} = [\overline{R}_1 \;\; \ldots \;\; \overline{R}_N]$ - expected asset reuturns

$\mathbf{S} = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1N} \\ \vdots & \ddots & \vdots \\ \sigma_{N1} & \cdots & \sigma_{NN} \end{bmatrix}$ - covariance matrix

$\mathbf{d} = [d_1 \;\; \ldots \;\; d_N]$ - asset dividend yields

$\mathbf{x} = [x_1 \;\; \ldots \;\; x_N]$ - portfolio weights