VERTICAL PLUME

CI 71Q HIDRODINAMICA AMBIENTAL   Profs. Y. Niño & A. Tamburrino
Sem. Otoño 2004

Consider the vertical discharge of a fluid of density ρ into an ambient stagnant fluid of uniform density ρ_0. If the discharged fluid is lighter than the ambient fluid ($\rho < \rho_0$), buoyancy effects associated to density differences between the two fluids tend to drive a vertical upwards motion denoted as *vertical plume* (Fig. 1).

In the vicinity of the discharge, the vertical motion is mostly driven by momentum, since the external fluid is usually discharged in the form of a jet. Nonetheless, if the momentum of the jet is rather low, buoyancy effects take over after a short distance from the discharge, and further away the fluid is driven mainly by buoyancy. The resulting plume is usually turbulent and this implies entrainment of ambient fluid into the upwards moving flow.

In what follows, the equations governing the flow in a circular vertical plume are developed. The analysis makes use of the boundary layer approximations, assuming that the plume radius, characterized by the length scale δ, is much smaller than a length scale characterizing the plume extension, L, such that: $\delta/L << 1$. Also invoked is the Boussinesq approximation, assuming the density difference between the discharged and ambient fluids is small.

1 Governing Equations

The relative density difference in this system is defined as:

$$\phi = \frac{\rho_0 - \rho}{\rho_0} = \frac{\Delta\rho}{\rho_0} = \phi(r, z) \tag{1}$$

which is a function of space, as indicated, where z and r denote vertical and radial coordinates, respectively. The problem is supposed to be axisymmetric so no dependence of ϕ on the horizontal angle, θ, exists, which is also valid for any of the variables characterizing fluid motion in the plume.

The instantaneous Navier-Stokes and conservation of volume and mass equations for steady axisymmetric flow, written in cylindrical coordinates and including Boussinesq approximation are:

$$u\,\frac{\partial w}{\partial r} + w\,\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\,\frac{\partial p}{\partial z} - (1-\phi)\,g + \nu\,\{\frac{1}{r}\,\frac{\partial}{\partial r}(r\,\frac{\partial w}{\partial r}) + \frac{\partial^2 w}{\partial z^2}\} \tag{2}$$

$$u\,\frac{\partial u}{\partial r} + w\,\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\,\frac{\partial p}{\partial r} + \nu\,\{\frac{1}{r}\,\frac{\partial}{\partial r}(r\,\frac{\partial u}{\partial r}) - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\} \tag{3}$$

$$\frac{\partial (r\,u)}{\partial r} + \frac{\partial (r\,w)}{\partial z} = 0 \tag{4}$$

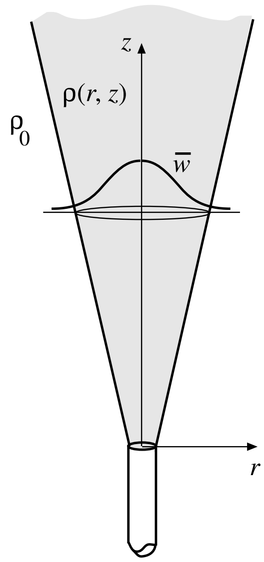

Figure 1: Circular vertical plume in stagnant ambient fluid of uniform density ρ_0.

$$\frac{\partial(r\ u\ \phi)}{\partial r} + \frac{\partial(r\ w\ \phi)}{\partial z} = D\ r\ \{\frac{1}{r}\ \frac{\partial}{\partial r}(r\ \frac{\partial \phi}{\partial r}) + \frac{\partial^2 \phi}{\partial z^2}\} \tag{5}$$

where (2) and (3) correspond to Navier-Stokes equations in the z and r directions, respectively, (4) corresponds to the conservation of volume equation and (5) corresponds to the conservation of mass equation. In these equations, u and w denote radial and vertical components of the flow velocity, p denotes thermodynamic pressure, g denotes acceleration of gravity, ν denotes kinematic viscosity of the fluid (assumed to be independent of ϕ as part of Boussinesq approximation) and D denotes the molecular diffusivity of the species causing the density difference associated with ϕ.

Introducing the typical decomposition of turbulent instantaneous variables into a mean and a fluctuation:

$$u = \bar{u} + u' \quad ; \quad w = \bar{w} + w' \quad ; \quad p = \bar{p} + p' \quad ; \quad \phi = \bar{\phi} + \phi' \tag{6}$$

replacing in the set of equations (2) to (5) and averaging over the turbulence, yields the Reynolds-averaged equations:

$$\bar{u}\ \frac{\partial \bar{w}}{\partial r} + \bar{w}\ \frac{\partial \bar{w}}{\partial z} = -\frac{1}{\rho_0}\ \frac{\partial \bar{p}}{\partial z} - (1-\bar{\phi})\ g + \nu\ \{\frac{1}{r}\ \frac{\partial}{\partial r}(r\ \frac{\partial \bar{w}}{\partial r}) + \frac{\partial^2 \bar{w}}{\partial z^2}\} - \frac{1}{r}\ \frac{\partial(r\ \overline{u'w'})}{\partial r} - \frac{\partial \overline{w'^2}}{\partial z} \tag{7}$$

$$\bar{u}\ \frac{\partial \bar{u}}{\partial r} + \bar{w}\ \frac{\partial \bar{u}}{\partial z} = -\frac{1}{\rho_0}\ \frac{\partial \bar{p}}{\partial r} + \nu\ \{\frac{1}{r}\ \frac{\partial}{\partial r}(r\ \frac{\partial \bar{u}}{\partial r}) - \frac{\bar{u}}{r^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\} - \frac{1}{r}\ \frac{\partial(r\ \overline{u'^2})}{\partial r} - \frac{\partial \overline{u'w'}}{\partial z} \tag{8}$$

$$\frac{\partial(r\ \bar{u})}{\partial r} + \frac{\partial(r\ \bar{w})}{\partial z} = 0 \tag{9}$$

$$\frac{\partial(r\ \bar{u}\ \bar{\phi})}{\partial r} + \frac{\partial(r\ \bar{w}\ \bar{\phi})}{\partial z} = D\ r\ \{\frac{1}{r}\ \frac{\partial}{\partial r}(r\ \frac{\partial \bar{\phi}}{\partial r}) + \frac{\partial^2 \bar{\phi}}{\partial z^2}\} - \frac{\partial(r\ \overline{u'\phi'})}{\partial r} - \frac{\partial(r\ \overline{w'\phi'})}{\partial z} \tag{10}$$

where the terms: $\overline{u'^2}$, $\overline{u'w'}$ and $\overline{w'^2}$ denote Reynolds turbulent stresses, while the terms: $\overline{u'\phi'}$ and $\overline{w'\phi'}$ denote turbulent mass fluxes in the radial and vertical directions, respectively.

Invoking now the boundary layer approximation, introducing the characteristic scales: U, W, δ and L, corresponding to the radial and vertical velocity components, and the radius and length of the plume, respectively, it can be assumed that:

$$\frac{U}{W} \approx \frac{\delta}{L} << 1 \quad ; \quad \frac{\partial}{\partial r} >> \frac{\partial}{\partial z} \tag{11}$$

scaling that can be used to reduce (7) to:

$$\bar{u}\,\frac{\partial \bar{w}}{\partial r} + \bar{w}\,\frac{\partial \bar{w}}{\partial z} = -\frac{1}{\rho_0}\,\frac{\partial \bar{p}}{\partial z} - (1-\bar{\phi})\,g - \frac{1}{r}\,\frac{\partial (r\,\overline{u'w'})}{\partial r} \tag{12}$$

where viscous terms have been neglected assuming that the Reynolds number of the flow is sufficiently large.

Considering that the ambient stagnant fluid has a hydrostatic pressure distribution and that the ambient pressure is imposed onto the boundary layer flow within the plume, then the vertical pressure gradient term can be reduced to:

$$\frac{1}{\rho_0}\,\frac{\partial \bar{p}}{\partial z} = -\rho_0\,g \tag{13}$$

which yields:

$$\bar{u}\,\frac{\partial \bar{w}}{\partial r} + \bar{w}\,\frac{\partial \bar{w}}{\partial z} = \bar{\phi}\,g + \frac{1}{r}\,\frac{\partial}{\partial r}(r\,\frac{\tau}{\rho_0}) \tag{14}$$

where $\tau = -\rho_0\,\overline{u'w'}$ denotes the vertical Reynolds shear stress. In the above equation, $\bar{\phi}\,g$ represents the buoyancy force, which is the main driving force of the plume flow. This term is the one that gives the plume a different behavior with respect to that of a simple jet with $\phi = 0$.

Applying the boundary layer approximation and neglecting viscous stresses in (8) yields:

$$\frac{\partial \bar{p}}{\partial r} = 0 \tag{15}$$

which indicates that the pressure remains constant in the radial direction, and therefore, invoking axial symmetry, in horizontal planes also.

Finally, the boundary layer approximation applied to (10) yields:

$$\frac{\partial (r\,\bar{u}\,\bar{\phi})}{\partial r} + \frac{\partial (r\,\bar{w}\,\bar{\phi})}{\partial z} = -\frac{\partial (r\,\overline{u'\phi'})}{\partial r} \tag{16}$$

where molecular diffusion has been neglected with respect to turbulent diffusion.

2 Cross-sectional Averaged Equations

The next step of the analysis consists of integrating the governing equations in the horizontal plane (r,θ), in the range: $0 < r < \infty$, $0 < \theta < 2\pi$.

Integrating (9), using Leibnitz rule, yields:

$$\frac{d}{dz}\int_0^\infty \bar{w}\,r\,dr = -(\bar{u}\,r)|_{r=\infty} \tag{17}$$

where the obvious boundary condition: $(\bar{u}\ r)|_{r=0} = 0$ has been used. The term: $-(\bar{u}\ r)|_{r=\infty}$ has a positive finite value and represents the entrainment of ambient fluid of density ρ_0 into the plume, which is manifested as a radial negative velocity at an infinite distance from the plume axis. Equation (17) multiplied by 2π represents the rate of variation in z of the volumetric discharge transported by the plume, which increases along z due to the entrainment process already discussed.

Integrating (16) using Leibnitz rule, leads to:

$$\frac{d}{dz}\, 2\pi \int_0^\infty \bar{w}\ \bar{\phi}\ r\ dr = 0 \tag{18}$$

where the boundary conditions: $(\bar{u}\ \bar{\phi}\ r)|_{r=0} = 0$, $\bar{\phi}|_{r=\infty} = 0$, $(\overline{u'\phi'}\ r)|_{r=0} = 0$ and $(\overline{u'\phi'})|_{r=\infty} = 0$ have been used, assuming that as $r \to \infty$, $\bar{\phi} \to 0$ and that the horizontal buoyancy flux at $r = \infty$ also vanishes. The latter condition is consistent with the idea that at $r = \infty$ there is only entrainment of ambient fluid with $\phi = 0$ into the plume.

From this result, it is readily obtained that:

$$2\pi \int_0^\infty \bar{w}\ \bar{\phi}\ r\ dr = constant \tag{19}$$

which indicates that the buoyancy flux transported by the plume remains constant along z. This flux is imposed externally as a boundary condition for the problem in analysis. Equation (19) simply express mass conservation in the system.

Multiplying (14) by r, adding continuity equation (9) and integrating in the (r, z) plane as before, using Leibnitz rule, yields:

$$\frac{d}{dz} \int_0^\infty \bar{w}^2\ r\ dr = g \int_0^\infty \bar{\phi}\ r\ dr \tag{20}$$

where the boundary conditions: $(\bar{w}\ \bar{\phi}\ r)|_{r=0} = 0$, $\bar{w}|_{r=\infty} = 0$, $(\bar{w}^2\ r)|_{r=0} = 0$, $(\tau\ r)|_{r=0} = 0$ and $\tau|_{r=\infty} = 0$ have been used, which consider that the entrainment velocity only has a radial component and that the Reynolds stress vanishes at a sufficiently large distance from the plume axis (that is, the turbulence decays to become negligible as $r \to \infty$).

Equation (20) indicates that the vertical momentum of the plume does not remain constant along z, because of the existence of $\bar{\phi} \neq 0$. On the contrary, the plume vertical momentum increases in z due to the effect of buoyancy. This is the main difference between a plume and a jet. In a jet, for which $\phi = 0$, the axial momentum is conserved.

3 Similarity Hypothesis

It is convenient now to introduce similarity hypotheses for the radial profiles of vertical velocity and relative density difference in the plume. These hypotheses eliminate the dependence on z of those profiles, imposing their collapse into curves that are independent of z when the proper scaling is used. With this aim, the scales W, Φ and b, are introduced for the axial mean velocity, the mean relative density difference and the radius of the plume, respectively. The following relationships are assumed to be valid:

$$\frac{\bar{w}}{W} = f(\eta) \quad ; \quad \frac{\bar{\phi}}{\Phi} = g(\eta) \quad ; \quad \eta = \frac{r}{b} \tag{21}$$

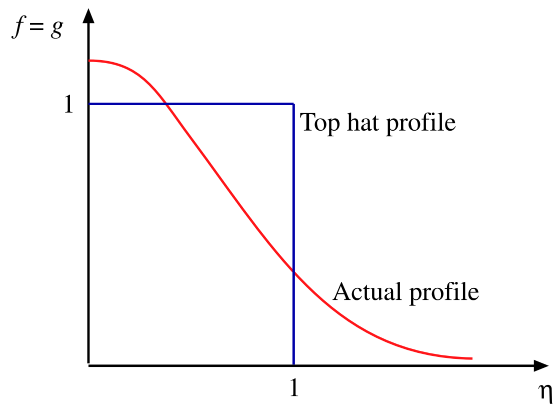

Figure 2: Top hat approximation for the similarity functions f and g.

where f and g are unknown functions of the dimensionless radius η.

If the functions f and g are effectively independent of z, then the following conditions must be valid, which can be used to define the variables W, Φ and b:

$$W^2\, b^2 = \frac{\int_0^\infty \bar{w}^2\, r\, dr}{S_1} \quad ; \quad W\, \Phi\, b^2 = \frac{\int_0^\infty \bar{w}\, \bar{\phi}\, r\, dr}{S_2} \quad ; \quad \Phi\, b^2 = \frac{\int_0^\infty \bar{\phi}\, r\, dr}{S_3} \tag{22}$$

where S_1, S_2 and S_3 represent shape factors, which are defined as:

$$S_1 = \int_0^\infty f^2\, \eta\, d\eta \quad ; \quad S_2 = \int_0^\infty f\, g\, \eta\, d\eta \quad ; \quad S_3 = \int_0^\infty g\, \eta\, d\eta \tag{23}$$

These factors can be only be evaluated from experimental data. However, by introducing the additional hypothesis of *top hat* profiles for f and g (as shown in Fig. 2) such that:

$$f(\eta) = g(\eta) = \begin{cases} 1 & 0 < \eta < 1 \\ 0 & \eta > 1 \end{cases} \tag{24}$$

then S_1, S_2 and S_3 are simply given by:

$$S_1 = S_2 = S_3 = \frac{1}{2} \tag{25}$$

With these approximations, (17), (19) and (20) can be reduced to:

$$\frac{d(W\, b^2)}{dz} = 2\, \alpha\, b\, W \tag{26}$$

$$2\, \pi\, W\, \Phi\, b^2 = constant \tag{27}$$

$$\frac{d(W^2\, b^2)}{dz} = g\, \Phi\, b^2 \tag{28}$$

where the entrainment velocity in (17) has been modeled as:

$$-(\bar{u}\, r)|_{r=\infty} = \alpha\, W\, b \tag{29}$$

with α denoting an entrainment coefficient. This model simply assumes that the entrainment velocity is proportional to the mean plume velocity.

Equation (26) indicates that the volumetric flow rate transported by the plume increases along z because of the entrainment of ambient fluid, equation (27) indicates that the mass flow rate transported by the plume is conserved, and equation (28) indicates that the z-momentum increases because of the presence of the relative density difference in the plume with respect to the ambient fluid.

In order to further simplify the governing equations for the plume, power laws for the variation of W, Φ and b with z are introduced, such that:

$$b \propto z^p \quad ; \quad W \propto z^q \quad ; \quad \Phi \propto z^t \tag{30}$$

where p, q and t are constants that can be deduced by replacing (30) in (26), (27) and (28). Indeed, it is easy to see just based on dimensional grounds that:

$$p = 1 \quad ; \quad q = -\frac{1}{3} \quad ; \quad t = -\frac{5}{3} \tag{31}$$

necessarily, which leads to the relationships:

$$b \propto z \quad ; \quad W \propto z^{-1/3} \quad ; \quad \Phi \propto z^{-5/3} \tag{32}$$

That is, the plume radius increases linearly as z increases, while the mean vertical velocity and mean relative density difference of the plume decrease as z increases. This behavior shows that due to mixing effects, the plume tends to increase in size, decreasing its velocity and increasing its dilution as it moves in the vertical.

Based on the previous result, the following relationships can be proposed:

$$b = b^* \, z \quad ; \quad W = W^* \, z^{-1/3} \quad ; \quad \Phi = \Phi^* \, z^{-5/3} \tag{33}$$

where b^*, W^* and Φ^* represent the coefficients of proportionality of the power laws for b, W and Φ, respectively. Their values can be obtained by replacing (33) in (26), (27) and (28). With that aim, the buoyancy flux (27) transported by the plume is denoted by F, which, as already discussed, is an invariant of the problem and an external parameter.

From (26):

$$\frac{d(W^* z^{-1/3} \, b^{*2} \, z^2)}{dz} = 2 \, \alpha \, b^* \, z \, W^* \, z^{-1/3} \tag{34}$$

which leads to:

$$b^* = \frac{6}{5} \, \alpha \tag{35}$$

Similarly, from (27):

$$2 \, \pi \, W^* \, z^{-1/3} \, \Phi^* \, z^{-5/3} \, b^{*2} \, z^2 = F \tag{36}$$

which gives:

$$W^* \, \Phi^* = \frac{25}{72} \, \frac{F}{\pi \alpha^2} \tag{37}$$

And Finally, from (28):

$$\frac{d(W^{*2}\ z^{-2/3}\ b^{*2}\ z^2)}{dz} = g\ \Phi^*\ z^{-5/3}\ b^{*2}\ z^2 \tag{38}$$

which yields:

$$\Phi^* = \frac{4}{3}\ \frac{W^{*2}}{g} \tag{39}$$

It is thus deduced that:

$$b^* = \frac{6}{5}\ \alpha \quad ; \quad W^* = (\frac{25}{96})^{1/3}\ (\frac{F\ g}{\pi\ \alpha^2})^{1/3} \quad ; \quad \Phi^* = \frac{1}{3}\ (\frac{25}{12})^{2/3}\ (\frac{F}{\pi\ \alpha^2\ g^{1/2}})^{2/3} \tag{40}$$

and replacing in (33) leads to:

$$b = 1.20\ \alpha\ z \tag{41}$$

$$W = 0.44\ (\frac{F\ g}{\alpha^2})^{1/3}\ z^{-1/3} \tag{42}$$

$$\Phi = 0.25\ (\frac{F}{\alpha^2\ g^{1/2}})^{2/3}\ z^{-5/3} \tag{43}$$

These expressions can be used to estimate the plume properties given the values of F and α. Nonetheless, while F is a parameter of the problem that is fixed externally, the entrainment coefficient α requires experimental data for its estimation.

4 Plume Dilution

The amount of dilution induced by entrainment of ambient fluid into the plume can be easily obtained from the previous results. This variable is of the upmost importance in the design of outfall discharges in lakes and coastal waters. Dilution is defined as the ratio between the volumetric discharge transported by the plume at a given z and the initial volumetric discharge. Calling S the dilution, Q_0 the initial volumetric discharge and $Q(z)$ the volumetric discharge of the plume at a height z, then, by definition:

$$S(z) = \frac{Q(z)}{Q_0} \tag{44}$$

where $Q(z)$ is given by:

$$Q(z) = 2\ \pi\ W\ b^2 \tag{45}$$

However, from (26), (41) and (42):

$$\frac{dQ}{dz} = 6.64\ (F\ g\ \alpha^4)^{1/3}\ z^{2/3} \tag{46}$$

equation that can be integrated between $z = 0$ (where $Q = Q_0$) and z to obtain:

$$Q(z) = Q_0 + 3.98\ (F\ g\ \alpha^4)^{1/3}\ z^{5/3} \tag{47}$$

Replacing (47) in (44) yields:

$$S = 1 + 3.98\ \frac{(F\ g\ \alpha^4)^{1/3}}{Q_0}\ z^{5/3} \tag{48}$$

which provides a relationship to estimate the dilution as a function of z in the vertical circular plume, which results to be a function of the buoyancy flux F, the entrainment coefficient, α, and the initial volumetric discharge, Q_0.

Empirically, it is known that the plume dilution is rather high (on the order of 100), which can be used to reduce (48) to:

$$S = 3.98\ \frac{(F\ g\ \alpha^4)^{1/3}}{Q_0}\ z^{5/3} \tag{49}$$

This equation can be compared with the empirical equation proposed by Fisher et al. (1979):

$$S = 0.15\ \frac{(F\ g)^{1/3}}{Q_0}\ z^{5/3} \tag{50}$$

Clearly, the form of equations (49) and (50) is identical, which validates the present analysis. By equating these two equations it is possible to estimate:

$$\alpha = 0.086 \tag{51}$$

value that is in close agreement with the result $\alpha = 0.083$, proposed by Fisher et al. (1979) for Gaussian distributions instead of the uniform distributions (*top hat* approximations) used in the present analysis for the velocity and relative density difference in the plume.

5 References

- Fisher, List, Koh, Imberger and Brooks. (1979). Mixing in Inland and Coastal Waters. Academic Press.