DENSITY CURRENTS

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Consider a fluid of density ρ, intruding along an inclined plane, with slope S, underneath a stagnant ambient fluid of lesser uniform density, ρ_0 (Fig. 1). The local density of the underflow or *density current* is given by:

$$\rho = \rho_0 \left(1 + \beta\right) \quad ; \quad \beta = \frac{\Delta\rho}{\rho_0} = \frac{(\rho - \rho_0)}{\rho_0} \tag{1}$$

where $\beta(x, z, t)$ is the instantaneous value of the relative density difference between the heavier fluid of the underflow or and the lighter ambient fluid. This variable fluctuates in time due to turbulence, nonetheless it is always positive, on the mean, for submerged density currents.

1 Governing Equations

If h is a measure of the thickness of the density current and H denotes the total depth of the ambient fluid, it is assumed in the following analysis that $h/H << 1$. With this assumption, boundary layer approximations are invoked to simplify the equations governing the motion of the density current. It is also assumed that β is small and the Boussinesq approximation can be used in the analysis.

With the Boussinesq approximation, the instantaneous equations governing fluid motion in the density current are:

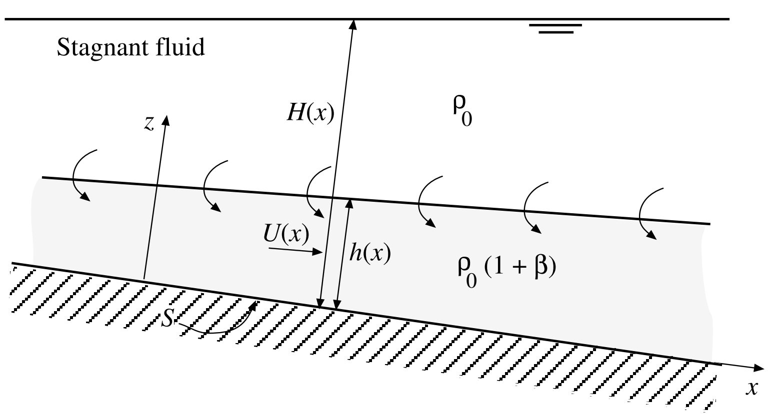


Figure 1: Submerged density current.

$$\rho_0 \{\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + w\,\frac{\partial u}{\partial z}\} = -\frac{\partial p}{\partial x} + \rho_0\,\nu\,\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\} + g\,\rho_0\,(1+\beta)\,S \tag{2}$$

$$\rho_0 \{\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial x} + w\,\frac{\partial w}{\partial z}\} = -\frac{\partial p}{\partial z} + \rho_0\,\nu\,\{\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\} - g\,\rho_0\,(1+\beta) \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{4}$$

$$\frac{\partial \beta}{\partial t} + \frac{\partial u\,\beta}{\partial x} + \frac{\partial w\,\beta}{\partial z} = D\,\{\frac{\partial^2 \beta}{\partial x^2} + \frac{\partial^2 \beta}{\partial z^2}\} \tag{5}$$

where (2) and (3) correspond to the momentum equations in directions x and z, respectively, (4) corresponds to the conservation of volume equation and (5) corresponds to the conservation of mass (or relative density difference of the density current) equation. Note that it is precisely β which drives the density current through buoyancy. In those equations u and w denote the instantaneous streamwise and bed normal velocity components, p denotes thermodynamic pressure, and ν denotes kinematic viscosity of the moving fluid.

The latter system of equations is next averaged over the turbulence. With that aim, each variable is decomposed into a mean value plus a fluctuation:

$$u = \bar{u} + u' \quad ; \quad w = \bar{w} + w' \quad ; \quad \beta = \bar{\beta} + \beta' \quad ; \quad p = \bar{p} + p' \tag{6}$$

With this decomposition, averaging over the turbulence reduces the previous system of equations to:

$$\rho_0 \{\frac{\partial \bar{u}}{\partial t} + \bar{u}\,\frac{\partial \bar{u}}{\partial x} + \bar{w}\,\frac{\partial \bar{u}}{\partial z}\} = -\frac{\partial \bar{p}}{\partial x} + \rho_0\,\nu\,\{\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\} + g\,\rho_0\,(1+\bar{\beta})\,S - \rho_0\,\frac{\partial \overline{u'^2}}{\partial x} - \rho_0\,\frac{\partial \overline{u'w'}}{\partial z} \tag{7}$$

$$\rho_0 \{\frac{\partial \bar{w}}{\partial t} + \bar{u}\,\frac{\partial \bar{w}}{\partial x} + \bar{w}\,\frac{\partial \bar{w}}{\partial z}\} = -\frac{\partial \bar{p}}{\partial z} + \rho_0\,\nu\,\{\frac{\partial^2 \bar{w}}{\partial x^2} + \frac{\partial^2 \bar{w}}{\partial z^2}\} - g\,\rho_0\,(1+\bar{\beta}) - \rho_0\,\frac{\partial \overline{w'^2}}{\partial z} - \rho_0\,\frac{\partial \overline{u'w'}}{\partial x} \tag{8}$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} = 0 \tag{9}$$

$$\frac{\partial \bar{\beta}}{\partial t} + \frac{\partial \bar{u}\,\bar{\beta}}{\partial x} + \frac{\partial \bar{w}\,\bar{\beta}}{\partial z} = D\,\{\frac{\partial^2 \bar{\beta}}{\partial x^2} + \frac{\partial^2 \bar{\beta}}{\partial z^2}\} - \frac{\partial \overline{u'\beta'}}{\partial x} - \frac{\partial \overline{w'\beta'}}{\partial z} \tag{10}$$

Where the terms $\overline{u'^2}$, $\overline{w'^2}$, $\overline{u'w'}$, represent Reynolds turbulent stresses, and the terms $\overline{u'\beta'}$ and $\overline{w'\beta'}$, represent Reynolds turbulent mass fluxes.

The next step consists of reducing the Reynolds averaged equations by introducing boundary layer approximations. For that, the following scaling is used:

$$\bar{u} \propto U \quad ; \quad z \propto h \quad ; \quad x \propto L \quad ; \quad \beta \propto B \tag{11}$$

where U denotes the mean velocity characteristic of the density current, h denotes the height of the current, L denotes a length scale in the streamwise direction, and B represents a mean value of the relative density difference of the current. The boundary layer approximation is obtained by invoking the condition: $h/L << 1$.

To further simplify the analysis, the assumption of a steady density current is considered.

From (9) it is concluded that

$$\frac{U}{L} \propto \frac{W}{h} \tag{12}$$

and therefore $W \propto U\ h/L$, where W is a measure of the bed normal velocity scale. Since $h/L << 1$ then it is obvious that W is of a lower magnitude than U.

Introducing the scales U, W, h and L, neglecting the term:

$$\frac{\overline{u'^2}}{U^2} << 1 \tag{13}$$

and taking the limits: $Re = U\ L/\nu \to \infty$ and $U\ h/\nu \to \infty$, in order to neglect small terms in the equations, it is possible to reduce (7) to:

$$\rho_0\ \{\bar{u}\ \frac{\partial \bar{u}}{\partial x} + \bar{w}\ \frac{\partial \bar{u}}{\partial z}\} = -\frac{\partial \bar{p}}{\partial x} + g\ \rho_0\ (1+\bar{\beta})\ S - \rho_0\ \frac{\partial \overline{u'w'}}{\partial z} \tag{14}$$

Repeating the same procedure for (8) yields the hydrostatic pressure approximation:

$$0 = -\frac{\partial \bar{p}}{\partial z} - g\ \rho_0\ (1+\bar{\beta}) \tag{15}$$

Doing a similar treatment of (10), neglecting molecular diffusion with respect to turbulent diffusion and neglecting the streamwise turbulent flux of β with respect to the corresponding streamwise advective flux, such that: $\overline{u'\beta'}/(U\ B) << 1$, and finally assuming that even though $\overline{w'\beta'}/(U\ B) << 1$ the corresponding vertical gradient of this variable is not negligible, yields:

$$\frac{\partial \bar{u}\ \bar{\beta}}{\partial x} + \frac{\partial \bar{w}\ \bar{\beta}}{\partial z} = -\frac{\partial \overline{w'\beta'}}{\partial z} \tag{16}$$

Integrating (15) in the vertical and imposing a zero value for the pressure at the free surface gives an expression for the pressure distribution of the flow:

$$\bar{p}(x,z) = \rho_0\ g\ \int_z^{H(x)} (1+\bar{\beta}(x,z))\ dz \tag{17}$$

Taking the derivative of this result with respect to x yields:

$$\frac{\partial \bar{p}}{\partial x} = \rho_0\ g\ S + \rho_0\ g\ \frac{\partial}{\partial x}\int_0^{\infty} \bar{\beta}\ dz \tag{18}$$

Where the relationship:

$$\frac{\partial H}{\partial x} = S \tag{19}$$

has been used, assuming a horizontal free surface, unperturbed by the submerged density current, and the upper limit of integration, H, has been replaced by the limit $z \to \infty$, assuming $H/h >> 1$.

2 Depth-averaged Equations

Next, the simplified equations of motion are integrated in the vertical.

Replacing (18) in (14) yields:

$$\rho_0 \{\bar{u}\,\frac{\partial \bar{u}}{\partial x} + \bar{w}\,\frac{\partial \bar{u}}{\partial z}\} = -\rho_0\,\frac{\partial}{\partial x}\int_z^\infty \bar{\beta}\,dz + \frac{\partial \tau}{\partial z} + g\,\rho_0\,\bar{\beta}\,S \tag{20}$$

where $\tau = -\rho_0 \overline{u'w'}$, represents the Reynolds shear stress in the x direction.

It is easy to see, using the conservation of volume equation (4) that:

$$\bar{u}\,\frac{\partial \bar{u}}{\partial x} + \bar{w}\,\frac{\partial \bar{u}}{\partial z} = \frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \bar{u}\bar{w}}{\partial z} \tag{21}$$

Replacing this result in (20) gives:

$$\rho_0 \{\frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \bar{u}\bar{w}}{\partial z}\} = -\rho_0\,\frac{\partial}{\partial x}\int_z^\infty \bar{\beta}\,dz + \frac{\partial \tau}{\partial z} + g\,\rho_0\,\bar{\beta}\,S \tag{22}$$

equation that can be depth-averaged to obtain:

$$\frac{\partial}{\partial x}\int_0^\infty \rho_0\,\bar{u}^2\,dz = -\rho\,g\frac{\partial}{\partial x}\int_0^\infty\int_z^\infty \bar{\beta}\,dz'dz - \tau_b + \rho_0\,g\,S\int_0^\infty \bar{\beta}\,dz \tag{23}$$

where the no-slip, no-penetration, no surface streamwise velocity and no surface shear stress boundary conditions have been imposed: $(\bar{u}\bar{w})|_{z=0} = 0$, $(\bar{u}\bar{w})|_{z=\infty} = 0$, $\tau|_{z=0} = \tau_b$, $\tau|_{z=\infty} = 0$, and τ_b denotes the bottom shear stress.

Integrating now (9) yields:

$$\frac{\partial}{\partial x}\int_0^\infty \bar{u}\,dz = \bar{w}_e \tag{24}$$

where the boundary conditions: $\bar{w}|_{z=0} = 0$ and $\bar{w}|_{z=\infty} = -\bar{w}_e$ have been used. Here $\bar{w}_e$ denotes the entrainment velocity or rate of entrainment of ambient fluid into the density current due to mixing effects.

Integrating (16) yields:

$$\frac{\partial}{\partial x}\int_0^\infty \bar{u}\,\bar{\beta}\,dz = 0 \tag{25}$$

where the zero surface relative density difference, no-penetration, no surface and no bottom mass flux boundary conditions: $\bar{\beta}|_{z=\infty} = 0$, $\bar{w}|_{z=0} = 0$, $\overline{w'\beta'}|_{z=\infty} = 0$, $\overline{w'\beta'}|_{z=0} = 0$, have been imposed.

The result (25) implies that the density current mass flux associated to β in direction x remains constant, and therefore it must be equal to the value of this flux at the origin of the density current ($U_0\,B_0\,h_0$), which is imposed externally:

$$\int_0^\infty \bar{u}\,\bar{\beta}\,dz = constant = U_0\,h_0\,B_0 \tag{26}$$

i.e., the flux $U_0\,B_0\,h_0$ is an invariant of the problem.

To continue with the analysis it is necessary to introduce further assumptions. In particular, closure relationships for the entrainment velocity, $\bar{w}_e$, and the bottom shear stress, τ_b are required. These are specified as:

$$\bar{w}_e = e_w \, U \tag{27}$$

$$\tau_b = \rho_0 \, c_f \, U^2 \tag{28}$$

where e_w denotes an entrainment coefficient and c_f a bottom friction coefficient.

It is necessary now to define more precisely the values of the thickness, h, the mean velocity, U, and the mean relative density difference, B, of the density current. These definitions can be expressed in terms of the following moments:

$$U \, h = \int_0^\infty \bar{u} \, dz \quad ; \quad U^2 \, h = \int_0^\infty \bar{u}^2 \, dz \quad ; \quad U \, B \, h = \int_0^\infty \bar{u} \, \bar{\beta} \, dz \tag{29}$$

where the first integral corresponds the volumetric flow discharge per unit width transported by the submerged current, the second integral corresponds to the streamwise momentum flux of the density current, and the third integral corresponds to the buoyancy flux transported by the density current in direction x.

From these integrals, U, h and B can be defined as:

$$U = \frac{U^2 \, h}{U \, h} = \frac{\int \bar{u}^2 \, dz}{\int \bar{u} \, dz} \quad ; \quad h = \frac{(U \, h)^2}{U^2 \, h} = \frac{(\int \bar{u} \, dz)^2}{\int \bar{u}^2 \, dz} \quad ; \quad B = \frac{U \, B \, h}{U \, h} = \frac{\int \bar{u} \, \bar{\beta} \, dz}{\int \bar{u} \, dz} \tag{30}$$

which all vary in the x direction.

3 Self Similar Solution

To further simplify the equations governing the motion of the density current, a self similarity hypothesis is introduced for the vertical distributions of $\bar{u}$ and $\bar{\beta}$. This similarity hypothesis reduces the dependence on x of the vertical distributions of $\bar{u}$ and $\bar{\beta}$, such that a collapse of them in one unique curve, when the variables are made dimensionless using the proper scaling, is possible.

Consider the following normalization:

$$\frac{\bar{u}}{U} = f_1(\eta) \quad ; \quad \frac{\bar{\beta}}{B} = f_2(\eta) \quad ; \quad \eta = \frac{z}{h} \tag{31}$$

Experimental results have shown that the curves f_1 and f_2 are indeed independent of x and they produce a satisfactory collapse of the experimental data (Parker et al., 1987).

Given the above definitions, the following conditions must hold if the proposed normalization is effective:

$$\int_0^\infty f_1(\eta) d\eta = 1 \quad ; \quad \int_0^\infty f_1^2(\eta) \, d\eta = 1 \quad ; \quad \int_0^\infty f_1(\eta) \, f_2(\eta) \, d\eta = 1 \tag{32}$$

Replacing (27), (28) and (30) in (23), (24) and (25) yields:

$$\frac{d(U^2 \, h)}{dx} = S_1 \, g \, B \, h \, S - \frac{1}{2} \, S_2 \, g \, \frac{d(B \, h^2)}{dx} - c_f \, U^2 \tag{33}$$

$$\frac{d(U \, h)}{dx} = e_w \, U \tag{34}$$

$$\frac{d(U\ B\ h)}{dx} = 0 \tag{35}$$

where S_1 and S_2 are *shape factors* defined by:

$$S_1 = \int_0^\infty f_2(\eta)\ d\eta \quad ; \quad S_2 = 2\int_0^\infty \int_\eta^\infty f_2(\eta)\ d\eta d\eta \tag{36}$$

which, as it is verified experimentally, have values close to unity.

From (35) it is concluded that the buoyancy flux is invariant along x, just as it was discussed previously:

$$U\ B\ h = U_0\ B_0\ h_0 = constant \tag{37}$$

where U_0, h_0 and B_0 denote the mean velocity, height and mean relative density difference, respectively, at the origin of the the density current, which corresponds to the boundary condition of the submerged flow.

Using the property of constant buoyancy flux, it is possible to manipulate equations (33) and (34) to obtain:

$$\frac{dh}{dx} = \frac{(2 - \frac{1}{2}\ S_2\ Ri)\ e_w - S_1\ Ri\ S + c_f}{(1 - S_2\ Ri)} \tag{38}$$

$$\frac{h}{3\ Ri}\ \frac{dRi}{dx} = \frac{(1 + \frac{1}{2}\ S_2\ Ri)\ e_w - S_1\ Ri\ S + c_f}{(1 - S_2\ Ri)} \tag{39}$$

where Ri denotes the *Richardson number* of the density current, defined as:

$$Ri = \frac{g\ B\ h}{U^2} \tag{40}$$

It is easy to see that Ri is the inverse of the *densimetric Froude number* of the flow.

The equations (38) and (39) were first proposed by Ellison and Turner in 1959. This set of equations, which is analogous to the equation for the computation of backwater curves in open channel flows, can be used to determine the gradually varied flow of submerged density currents. Moreover, by neglecting e_w in the above equations the result obtained in a previous set of notes for non-mixing underflows is recovered.

In non-mixing density currents (or even in open channel flows), h and Ri are linked through the continuity equation. Since in such case there is no entrainment of ambient fluid into the underflow, (34) gives $U\ h = q = constant$, where q denotes the volumetric discharge per unit width. The Richardson number in this case results to be proportional to h^3. This leads to a uniform flow situation for which $dh/dx = 0$, that is, to a flow in which h is constant and therefore Ri is also constant. In the case of density currents, however, the volumetric discharge $U\ h$ increases along x due to entrainment of ambient fluid. In such case, B varies along the current and therefore Ri varies with both h and B. As a result, no uniform flow situation ($dh/dx = 0$) is possible in the case of density currents.

It is important to note that the use of (38) and (39) requires knowledge about the shape factors, S_1 and S_2. Those can only be obtained from experimental observations.

The entrainment coefficient, e_w, is, in general, a function of the Richardson number of the flow. This coefficient can be evaluated using the relationship proposed by García (1985):

$$e_w = \frac{0.075}{(1 + 715\ Ri^{2.4})^{0.5}} \tag{41}$$

It is easy to see that when $Ri \to 0$, $e_w \to 0.075$, which is the value corresponding to a wall plane jet (that is, to a flow without stratification). For large values of Ri, (41) can be approximated by:

$$e_w = 2.8 \times 10^{-3}\ Ri^{-1.2} \tag{42}$$

4 Analysis of gradually varied flow types in density currents

In this section the possible backwater curves corresponding to submerged density currents are analyzed. As it is shown next, these backwater curves are somewhat similar to those of open channel flow, however due to the entrainment of ambient fluid into the underflow, the number of possible situations is larger and more complex flow situations are possible.

4.1 Governing equations

The already obtained equations governing the motion of density currents can be rewritten as:

$$\frac{dh}{dx} = \frac{(2\ e_w + c_f) - (S_1\ S + \frac{1}{2}\ S_2\ e_w)\ Ri}{(1 - S_2\ Ri)} \tag{43}$$

$$\frac{h}{3\ Ri}\ \frac{dRi}{dx} = \frac{(e_w + c_f) - (S_1\ S - \frac{1}{2}\ S_2\ e_w)\ Ri}{(1 - S_2\ Ri)} = \frac{dh}{dx} - e_w \tag{44}$$

$$\frac{dU}{dx} = -\frac{U}{3\ Ri}\ \frac{dRi}{dx} \tag{45}$$

$$\frac{1}{B}\ \frac{dB}{dx} = -\frac{e_w}{h} \tag{46}$$

$$e_w = f(Ri) \tag{47}$$

The last equation simply indicates that the entrainment coefficient is a function of the Richardson number of the flow. According to (41) e_w always decreases as Ri increases.

4.2 Characteristic values of the Richardson number

From the analysis of the above system of equations, the following characteristic values of the Richardson number are deduced:

- Critical flow

  This condition occurs when:

$$\frac{dh}{dx} \to \pm\infty \tag{48}$$

Therefore, from (43) the critical Richardson number of the flow is given by:

$$Ri_c = \frac{1}{S_2} \tag{49}$$

and the following flow conditions are defined: $Ri > Ri_c$ corresponds to *subcritical flow*, and $Ri < Ri_c$ corresponds to *supercritical flow*.

- Normal or equilibrium flow

  This condition occurs when:

$$\frac{dRi}{dx} = 0 \tag{50}$$

that is, when $Ri = Ri_n = constant$, where Ri_n represents the *normal* value of the Richardson number. Replacing in (44) yields:

$$Ri_n = \frac{e_{wn} + c_f}{S_1\ S - \frac{1}{2}\ S_2\ e_{wn}} \tag{51}$$

where e_{wn} represents the entrainment coefficient associated to the normal flow given by:

$$e_{wn} = f(Ri_n) = constant \tag{52}$$

It is important to note that since, from a physical point of view, the Richardson number is always positive, from (51) it is concluded that the following condition must always hold:

$$S_1\ S > \frac{1}{2}\ S_2\ e_{wn} \tag{53}$$

For these conditions, from (43) it is concluded that the normal flow depth h_n is given by:

$$\frac{dh_n}{dx} = e_{wn} \tag{54}$$

which implies that h_n increases linearly with the distance x, due to entrainment of ambient fluid into the density current, according to the equation:

$$h(x) = h_0 + e_{wn}\ (x - x_0) \tag{55}$$

where h_0 represents a boundary condition for the integration, imposed at $x = x_0$.

Replacing (50) in (45) yields:

$$\frac{dU}{dx} = 0 \tag{56}$$

which indicates that for normal flow conditions both the Richardson number and the mean velocity of the underflow are constant. This last condition gives: $U = U_n = constant$, where U_n denotes the normal flow velocity. Interestingly, this results occurs as the flow depth increases along x.

Finally, (46) implies that the mean relative density difference varies as:

$$B_n = \frac{Ri_n\, U_n^2}{g\, h_n(x)} \tag{57}$$

which from (55) implies that B_n decreases along x and that the current tends to disappear due to dilution created by the entrainment of ambient fluid into the underflow.

- Uniform flow

  This condition occurs when:

$$\frac{dh}{dx} = 0 \tag{58}$$

Replacing this condition in (43) yields the Richardson number associated with the uniform flow:

$$Ri_u = \frac{2\, e_{wu} + c_f}{S_1\, S - \frac{1}{2}\, S_2\, e_{wu}} \tag{59}$$

However, from (47) it is concluded that (58) implies:

$$\frac{dRi}{dx} = -e_{wu}\, \frac{3\, Ri_u}{h_u} \leq 0 \tag{60}$$

which indicates that the Richardson number of this flow cannot remain constant, equal to Ri_u, unless $e_w = 0$. In other words, uniform flow depth density currents are not possible, unless mixing with the ambient fluid is negligible.

4.3 Gradually varied density current types

With the aim of analyzing the possible different flow situations associated with density currents, it is convenient to introduce the following definitions:

$$Ri_1 = \frac{2\, e_w + c_f}{S_1\, S + \frac{1}{2}\, S_2\, e_w} \tag{61}$$

$$Ri_2 = \frac{e_w + c_f}{S_1\, S - \frac{1}{2}\, S_2\, e_w} \tag{62}$$

Both Ri_1 and Ri_2 are functions of the Richardson number of the flow. Moreover, from (51) and (59) it can be shown that:

$$Ri_1(Ri_u) = Ri_u \tag{63}$$

$$Ri_2(Ri_n) = Ri_n \tag{64}$$

Introducing the definitions (61) and (62) in equations (43) and (44) yields:

$$\frac{dh}{dx} = \frac{S_1\, S + \frac{1}{2}\, S_2\, e_w}{S_2}\, \frac{(Ri_1 - Ri)}{(Ri_c - Ri)} \tag{65}$$

$$\frac{h}{3\, Ri}\, \frac{dRi}{dx} = \frac{S_1\, S - \frac{1}{2}\, S_2\, e_w}{S_2}\, \frac{(Ri_2 - Ri)}{(Ri_c - Ri)} = \frac{dh}{dx} - e_w \tag{66}$$

With the aim of analyzing the sign of the gradients of h and Ri of equations (65) and (66), it is convenient to note that the following relationships are valid:

$$Ri < Ri_c \quad \rightarrow \quad Ri_n < Ri_u \tag{67}$$

$$Ri > Ri_c \quad \rightarrow \quad Ri_n > Ri_u \tag{68}$$

$$Ri < Ri_n \quad \rightarrow \quad Ri_2 > Ri \tag{69}$$

$$Ri > Ri_n \quad \rightarrow \quad Ri_2 < Ri \tag{70}$$

$$Ri < Ri_n \quad \rightarrow \quad e_w > e_{wn} \tag{71}$$

$$Ri > Ri_n \quad \rightarrow \quad e_w < e_{wn} \tag{72}$$

$$Ri < Ri_u \quad \rightarrow \quad Ri_1 > Ri \tag{73}$$

$$Ri > Ri_u \quad \rightarrow \quad Ri_1 < Ri \tag{74}$$

The classification of the possible flow types is done based on the value of Ri relative to the three characteristic values of the Richardson number in density currents, Ri_c, Ri_n, Ri_u. The bottom slope is defined as *mild* if:

$$Ri_c < Ri < Ri_n \tag{75}$$

and as *steep* if:

$$Ri_n < Ri < Ri_c \tag{76}$$

According to this classification, eight different cases of backwater curves are obtained for the density currents in analysis, which are summarized in Figs. 2 and 3. Those cases have been identified in the figures, in terms of the sign of the gradients of h and Ri in each case, which are indicated with the signs "+" (indicating that the gradient is positive) and "−" (indicating that the gradient is negative).

5 References

- Ellison, T.H. and Turner, J.S. (1959) Turbulent entrainment in stratified flows. Journal of Fluid Mechanics. Vol. 6, pp. 423-448.
- García, M. (1985) Experimental study of turbidity currents. M.Sc. Thesis. University of Minnesota.
- Parker, G., García, M., Fukushima, Y. and Yu, W. (1987) Experiments on turbidity currents over an erodible bed. Journal of Hydraulic Research, 25 (1), pp. 123-147.

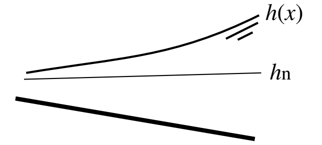
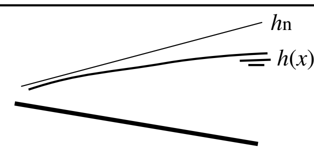
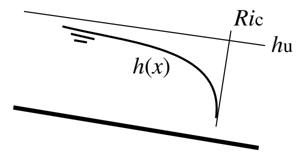
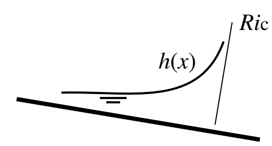
Case	d Ri/d x	d h/d x	Backwater type
$Ric < Riu < Rin < Ri$	+	+	
$Ric < Riu < Ri < Rin$	−	+	
$Ric < Ri < Riu < Rin$	−	−	
$Ri < Ric < Riu < Rin$	+	+	

Figure 2: Density currents in Mild slope ($Ri_c < Ri_u < Ri_n$).

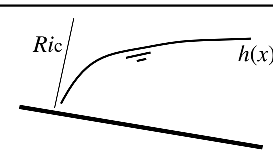
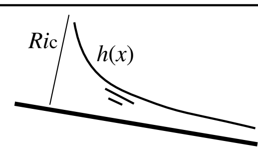
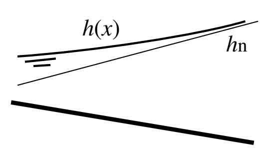
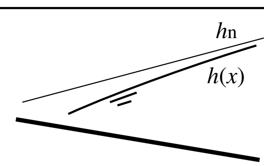
Case	d Ri/d x	d h/d x	Backwater type
$Rin < Riu < Ric < Ri$	+	+	
$Rin < Riu < Ri < Ric$	−	−	
$Rin < Ri < Riu < Ric$	−	+	
$Ri < Rin < Riu < Ric$	+	+	

Figure 3: Density currents in Steep slope ($Ri_n < Ri_u < Ri_c$).