

SALT WEDGE

CI 71Q HIDRODINAMICA AMBIENTAL Profs. Y. Niño & A. Tamburrino
Sem. Otoño 2004

Fresh water rivers and streams discharging into the ocean are usually affected by salinity intrusion. In these estuarial systems, salt water from the ocean tends to move upstream, driven by buoyancy created by density differences between fresh and salt water. Depending on mixing conditions in the estuary, salinity can be totally or partially mixed in the water column, or not mixed at all, if a two-layer stratified flow situation occurs, as shown in Fig. 1. In the latter case, called a *salt wedge* intrusion, there is little to no mixing at the density interface between the salt water bottom layer and the freshwater upper layer. Since the less dense freshwater continues moving towards the ocean, it goes over the salt wedge, which acts basically as a front that intrudes into the fresh water estuary. However, as the shear stress acting along the density interface balances out the effect of gravity, the salinity front eventually stops moving upstream and a steady state situation is reached, at which the salt wedge is said to be *arrested*, reaching a total length L measured from the river mouth upstream.

As shown in Fig. 1, the flow in the salt wedge can be represented as a two-layer stratified flow. In the upper layer the freshwater moves in the direction of the ocean, while in the bottom layer the salt water recirculates, moving in the direction of the ocean in the top region of the layer and in the opposite direction, upstream with respect to the freshwater discharge, near the bottom. Because the salt wedge is arrested, the net salt water discharge in the bottom layer is zero.

The upstream length of the salinity intrusion depends mainly on the freshwater discharge and the slope of the estuary, S_0 . Large values of the discharge tends to move the salt wedge towards the ocean. On the contrary, during periods of low freshwater discharge, the salt wedge tends to intrude for rather long distances upstream. In such circumstances, salt wedges have been observed to intrude distances as large as 50 to 100 km upstream from the ocean.

Even under steady freshwater discharges, tidal effects can create a back and forth motion of the salt wedge, with a period equal to that of the forcing tide. Large excursions of the salt wedge in response to tidal forcing can induce strong mixing, in some cases enhanced by interactions with the bathymetry of the estuary. In such cases, the stratification can be weakened enough as to destroy the two-layer structure of the salt wedge. Accordingly, such cases are classified as well- or partially-mixed estuaries. Fisher et al. (1979) propose a criterion based on an *estuary Richardson number*, defined as:

$$Ri_e = \frac{\Delta\rho g Q_f}{\rho_s W u_t^3}$$

where $\Delta\rho = \rho_s - \rho_0$ is the density difference between salt and fresh water, with ρ_s and ρ_0 denoting salt and fresh water density, respectively, g denotes acceleration of gravity, Q_f denotes the total

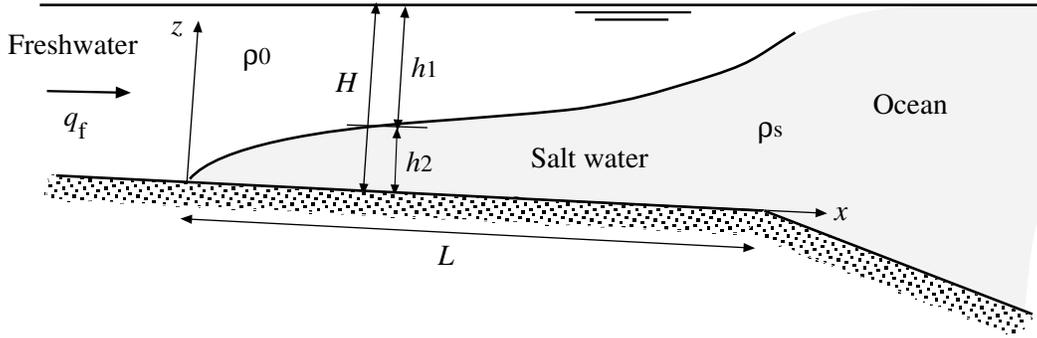


Figure 1: Arrested salt wedge.

freshwater discharge, W denotes the estuary width, and u_t denotes the root-mean-square velocity in the estuary influenced by tidal effects. If $Ri_e > 0.8$, the estuary is considered to be strongly stratified and the flow dominated by a salt wedge intrusion. If $Ri_e < 0.08$, the estuary is considered to be well mixed and the vertical density variation in the cross section negligible. Other criteria for estuary classification in terms of mixing characteristics can be found in Martin and McCutcheon (1999).

1 Governing equations

Using boundary layer approximations and averaging over the turbulence, the governing equations for the flow situation under consideration are:

$$\rho_0 (1 + \phi) \left\{ u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right\} = - \frac{\partial p}{\partial x} + \rho_0 (1 + \phi) g S_0 + \frac{\partial \tau}{\partial z} \quad (1)$$

$$- \frac{\partial p}{\partial z} - \rho_0 (1 + \phi) g = 0 \quad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad (3)$$

$$\frac{\partial(u \phi)}{\partial x} + \frac{\partial(w \phi)}{\partial z} = - \frac{\partial F_\phi}{\partial z} \quad (4)$$

where (1) represents the momentum equation in x , (2) represents the momentum equation in z , (3) represents the equation of conservation of volume assuming incompressible fluid and (4) represents the equation of conservation of mass of salt. In the system of equations, u , w , and p denote flow velocities in x and z and thermodynamic pressure, respectively, all averaged over the turbulence.

In (1) the total shear stress, τ is given by viscous and Reynolds stresses:

$$\tau = -\rho_0 (1 + \phi) \overline{u'w'} + \mu \frac{\partial u}{\partial z} \quad (5)$$

In (4), F_ϕ denotes the mean vertical flux of relative density difference due to the presence of salt, induced by both molecular and turbulent diffusion, which is given by:

$$F_\phi = \overline{\phi'w'} - D \frac{\partial \phi}{\partial z} \quad (6)$$

where D denotes the molecular diffusivity of salt in water.

2 Layer-averaged equations

The volumetric fresh water flow discharge per unit width in the upper layer is given by:

$$q_f = \int_{h_2}^H u \, dz \quad (7)$$

So, defining the thickness of the bottom layer, h_2 , such that:

$$\int_0^{h_2} u \, dz = 0 \quad (8)$$

and integrating (3) in the bottom layer defined by $0 < z < h_2$ (Fig. 1),

$$\int_0^{h_2} \frac{\partial u}{\partial x} \, dz + \int_0^{h_2} \frac{\partial w}{\partial z} \, dz = 0 = \frac{d}{dx} \int_0^{h_2} u \, dz - u|_{z=h_2} \frac{dh_2}{dx} + w|_{z=h_2} \quad (9)$$

where Leibnitz integration rule was invoked, yields:

$$u|_{z=h_2} \frac{dh_2}{dx} - w|_{z=h_2} = 0 \quad (10)$$

Integrating (3) again, but this time in the upper layer ($h_2 < z < H$) and using (10), it is possible to show that the flow discharge q_f remains constant along x :

$$\frac{d}{dx} \int_{h_2}^H u \, dz = \frac{dq_f}{dx} = 0 \quad (11)$$

Integrating (4) in the upper layer, imposing the condition:

$$F_\phi|_{z=H} = 0 \quad (12)$$

which implies that the mass flux of salt through the free surface is zero, and from (10):

$$-(u \phi)|_{z=h_2} \frac{dH}{dx} + (w \phi)|_{z=h_2} = 0 = (u \phi)|_{z=h_2} \frac{dh_2}{dx} - (w \phi)|_{z=h_2} \quad (13)$$

gives:

$$\frac{d}{dx} \int_{h_2}^H u \phi \, dz = F_{\phi i} \quad (14)$$

where:

$$F_{\phi i} = F_\phi|_{z=h_2} \quad (15)$$

represents the vertical mass flux of salt across the density interface between upper and bottom layers.

Analogously, integrating (4) now in the bottom layer and imposing:

$$F_\phi|_{z=0} = 0 \quad (16)$$

which implies that the vertical mass flux of salt through the bottom wall is zero, yields:

$$\frac{d}{dx} \int_0^{h_2} u \phi dz = -F_{\phi i} \quad (17)$$

In the case of a typical salt wedge, the effect of the strong stratification usually precludes mixing at the density interface. Accordingly, the following simple assumption is used in what follows:

$$F_{\phi i} \approx 0 \quad (18)$$

In order to determine the pressure gradient in x , (2) is integrated vertically, imposing a vanishing value of the relative thermodynamic pressure at the free surface. This yields:

$$p = \rho_0 g \left\{ (H - z) + \int_z^H \phi dz' \right\} \quad (19)$$

which is valid in both the upper and bottom layers of the stratified flow in analysis. The pressure gradient in x is therefore given by:

$$-\frac{\partial p}{\partial x} = -\rho_0 g \left\{ \frac{dH}{dx} + \frac{\partial}{\partial x} \int_z^H \phi dz' \right\} \quad (20)$$

Replacing this equation in (1) and dividing by ρ_0 yields:

$$(1 + \phi) \left\{ u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right\} = -g \left\{ \frac{dH}{dx} + \frac{\partial}{\partial x} \int_z^H \phi dz' \right\} + \rho_0 (1 + \phi) g S_0 + \frac{1}{\rho_0} \frac{\partial \tau}{\partial z} \quad (21)$$

The left hand side of this equation can be transformed as follows:

$$\begin{aligned} (1 + \phi) \left\{ u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right\} &= (1 + \phi) \left\{ \frac{\partial u^2}{\partial x} + \frac{\partial(u w)}{\partial z} \right\} = \frac{\partial}{\partial x} \left\{ (1 + \phi) u^2 \right\} + \frac{\partial}{\partial z} \left\{ (1 + \phi) u w \right\} - \\ &u^2 \frac{\partial \phi}{\partial x} - u w \frac{\partial \phi}{\partial z} \end{aligned} \quad (22)$$

but from (3) and (4):

$$u \frac{\partial \phi}{\partial x} + w \frac{\partial \phi}{\partial z} = -\frac{\partial F_\phi}{\partial z} \quad (23)$$

hence:

$$(1 + \phi) \left\{ u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right\} = \frac{\partial}{\partial x} \left\{ (1 + \phi) u^2 \right\} + \frac{\partial}{\partial z} \left\{ (1 + \phi) u w \right\} + u \frac{\partial F_\phi}{\partial z} \quad (24)$$

Now, with this result, integrating each term of (21) in the upper layer gives, respectively:

$$\int_{h_2}^H (1 + \phi) \left\{ u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right\} dz = \frac{d}{dx} \int_{h_2}^H (1 + \phi) u^2 dz - \int_{h_2}^H u \frac{\partial F_\phi}{\partial z} dz \quad (25)$$

$$-g \int_{h_2}^H \left\{ \frac{dH}{dz} + \frac{\partial}{\partial x} \int_{z'}^H \phi dz'' \right\} dz' = -g h_1 \frac{dH}{dx} - g \int_{h_2}^H \frac{\partial}{\partial x} \int_{z'}^H \phi dz'' dz' \quad (26)$$

$$g S_0 \int_{h_2}^H (1 + \phi) dz = g S_0 \left\{ (H - h_2) + \int_{h_2}^H \phi dz \right\} \quad (27)$$

$$\frac{1}{\rho_0} \int_{h_2}^H \frac{\partial \tau}{\partial z} dz = \frac{1}{\rho_0} (\tau|_H - \tau|_{h_2}) = -\frac{\tau_i}{\rho_0} \quad (28)$$

where τ_i denotes the shear stress at the density interface: $\tau_i = \tau|_{z=h_2}$.

Considering that from (21): (25) = (26) + (27) + (28), the following result is obtained for the upper layer:

$$\begin{aligned} \frac{d}{dx} \int_{h_2}^H (1 + \phi) u^2 dz - \int_{h_2}^H F_\phi \frac{\partial u}{\partial z} dz = -g \left\{ h_1 \frac{dH}{dx} + \int_{h_2}^H \frac{\partial}{\partial x} \int_{z'}^H \phi dz'' dz' \right\} + \\ g S_0 \left\{ (H - h_2) + \int_{h_2}^H \phi dz \right\} - \frac{\tau_i}{\rho_0} \end{aligned} \quad (29)$$

Doing the same vertical integral of each term of (21) as before, but now in the bottom layer yields, respectively:

$$\int_0^{h_2} (1 + \phi) \left\{ u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right\} dz = \frac{d}{dx} \int_0^{h_2} (1 + \phi) u^2 dz - \int_0^{h_2} u \frac{\partial F_\phi}{\partial z} dz \quad (30)$$

$$-g \int_0^{h_2} \left\{ \frac{dH}{dz} + \frac{\partial}{\partial x} \int_{z'}^H \phi dz'' \right\} dz' = -g h_2 \frac{dH}{dx} - g \int_0^{h_2} \frac{\partial}{\partial x} \int_{z'}^H \phi dz'' dz' \quad (31)$$

$$g S_0 \int_0^{h_2} (1 + \phi) dz = g S_0 \left\{ h_2 + \int_0^{h_2} \phi dz \right\} \quad (32)$$

$$\frac{1}{\rho_0} \int_0^{h_2} \frac{\partial \tau}{\partial z} dz = \frac{1}{\rho_0} (\tau|_{h_2} - \tau|_0) = \frac{\tau_i + \tau_0}{\rho_0} \quad (33)$$

where τ_0 denotes the bottom shear stress: $\tau_0 = \tau|_{z=0}$.

Again, considering that from (21): (30) = (31) + (32) + (33), the following result is obtained for the bottom layer:

$$\begin{aligned} \frac{d}{dx} \int_0^{h_2} (1 + \phi) u^2 dz - \int_0^{h_2} F_\phi \frac{\partial u}{\partial z} dz = -g \left\{ h_2 \frac{dH}{dx} + \int_0^{h_2} \frac{\partial}{\partial x} \int_{z'}^H \phi dz'' dz' \right\} + \\ g S_0 \left\{ h_2 + \int_0^{h_2} \phi dz \right\} - \frac{(\tau_i + \tau_0)}{\rho_0} \end{aligned} \quad (34)$$

The negative sign chosen for τ_0 responds to the need of using the convention $\tau_0 > 0$ to represent a bottom shear stress that opposes the fluid motion in the nearby region, within the salt water layer, which in this case is in the upstream direction with respect to the fresh water discharge.

Fig. 2 shows vertical profiles of u and ϕ in a salt wedge, observed in a laboratory experiment. Flow velocities in the bottom layer are much lower than those of the freshwater flow of the upper layer. Given the definition of h_2 in (8), chosen just to let the net salt water discharge to vanish in

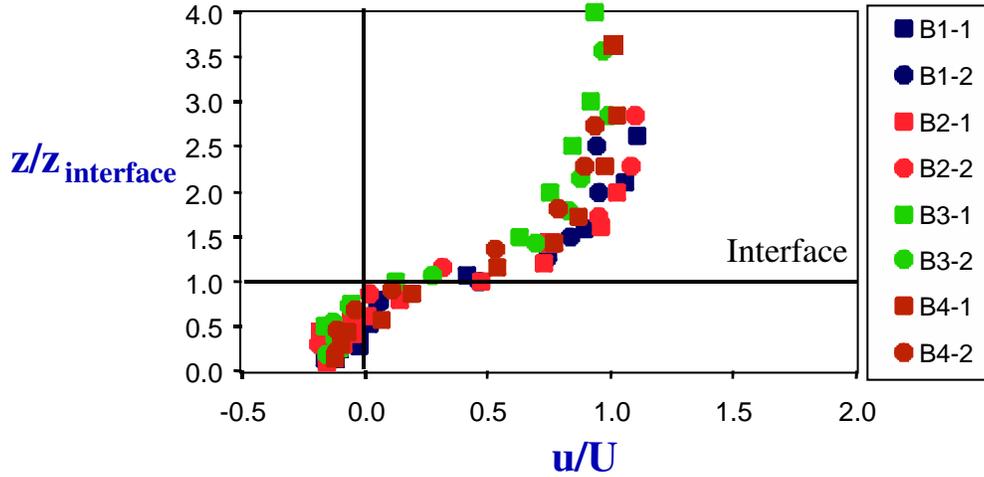


Figure 2: Vertical velocity profiles in the salt wedge. Experimental results show self similarity along the salt wedge and small dependence on the densimetric Froude number of the flow

the bottom layer, it is easy to see that the following approximation reproduces, at least at a coarse level, the observed velocity profile:

$$u(x, z) = \begin{cases} U(x) & ; z \geq h_2 \\ 0 & ; z < h_2 \end{cases} \quad (35)$$

This is known as *top hat* approximation. The vertical structure of ϕ can be approximated similarly as:

$$\phi(x, z) = \begin{cases} \Phi_1(x) & ; z \geq h_2 \\ \Phi_2(x) & ; z < h_2 \end{cases} \quad (36)$$

Replacing (35) in (7) yields:

$$q_f = \int_{h_2}^H u dz = U h_1 \quad (37)$$

On the other hand, replacing (35) and (36) in (14) and using (18) yields:

$$\int_{h_2}^H u \phi dz = U \Phi_1 h_1 = 0 \quad (38)$$

which implies that:

$$q_f \Phi_1 = 0 \quad (39)$$

and since $q_f \neq 0$, then it is concluded that $\Phi_1 = 0$.

Similarly, replacing (35) and (36) in (29), with $\Phi_1 = 0$ gives:

$$\frac{d(U^2 h_1)}{dx} = -g h_1 \frac{dH}{dx} + g S_0 h_1 - \frac{\tau_i}{\rho_0} \quad (40)$$

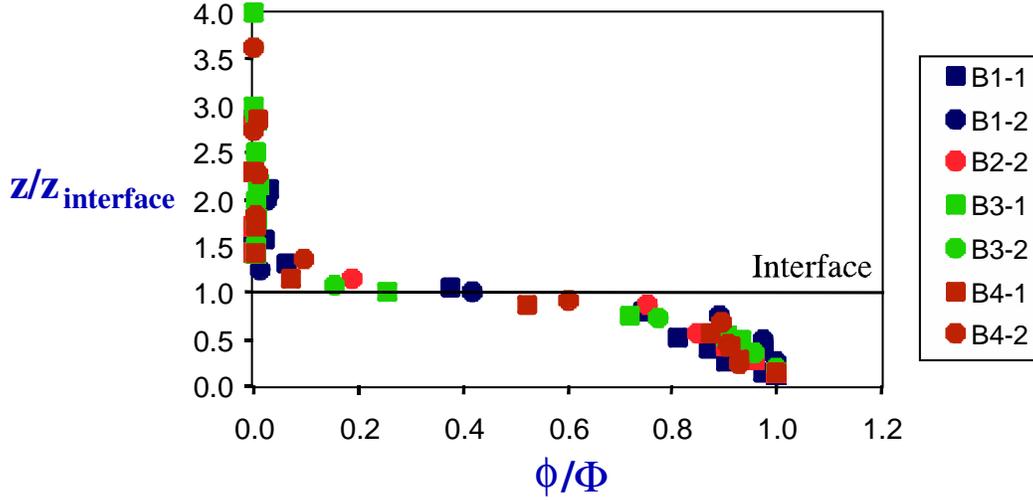


Figure 3: Vertical relative density difference profiles in the salt wedge. Experimental results show self similarity along the salt wedge and small dependence on the densimetric Froude number of the flow

but from (37):

$$U^2 h_1 = \frac{q_f^2}{h_1} \quad (41)$$

then, replacing in (40) yields:

$$\frac{d(q_f^2/h_1)}{dx} = -g h_1 \frac{dH}{dx} + g S_0 h_1 - \frac{\tau_i}{\rho_0} \quad (42)$$

which represents the equation governing the streamwise momentum change in the x direction, in the upper layer. The first term of the right hand side corresponds to the net pressure force acting on the upper layer, the second one corresponds to the gravity force acting on the upper layer, projected in the streamwise direction, and the third one corresponds to the friction force acting on the upper layer, exerted by the interfacial shear stress.

Replacing (35) and (36) in (34) and calling $\Phi_2 = \Phi$ yields:

$$-g \left\{ h_2 \frac{dH}{dx} + B_* \right\} + g S_0 h_2 (1 + \Phi) + \frac{\tau_i + \tau_0}{\rho_0} = 0 \quad (43)$$

where:

$$B_* = \int_0^{h_2} \frac{\partial}{\partial x} \int_{z'}^H \phi dz'' dz' = \Phi \int_0^{h_2} \frac{\partial}{\partial x} \int_{z'}^{h_2} dz'' dz' = \Phi \int_0^{h_2} \frac{\partial}{\partial x} (h_2 - z') dz' = \Phi h_2 \frac{dh_2}{dx} \quad (44)$$

Replacing this result in (43) finally gives:

$$-g \left\{ h_2 \frac{dH}{dx} + \Phi_2 h_2 \frac{dh_2}{dx} \right\} + g (1 + \Phi) h_2 S_0 + \frac{\tau_i + \tau_0}{\rho_0} = 0 \quad (45)$$

In most applications, the flow velocities within the bottom layer, in the vicinity of the bottom wall, are so small that the bottom shear stress, τ_0 , results to be negligible compared with the interfacial shear stress, τ_i . On the other hand, the mean bottom slope of estuaries is frequently very small, such that S_0 can also be neglected in (42) and (45). With these approximations the latter set of equations is reduced to:

$$\left(-\frac{q_f^2}{g h_1^3} + 1\right) \frac{dh_1}{dx} = -\frac{dh_2}{dx} - \frac{\tau_i}{\rho_0 g h_1} \quad (46)$$

$$\frac{dh_2}{dx} = \left(-\frac{dh_1}{dx} + \frac{\tau_i}{\rho_0 g h_2}\right) \left(\frac{1}{1 + \Phi}\right) \approx (1 - \Phi) \left(-\frac{dh_1}{dx} + \frac{\tau_i}{\rho_0 g h_2}\right) \quad (47)$$

where the relationship: $H = h_1 + h_2$ has been invoked.

Replacing (47) in (46) and using the following resistance closure for the interfacial shear stress:

$$\tau_i = \rho_0 c_{fi} U^2 = \rho_0 c_{fi} \frac{q_f^2}{h_1^2} \quad (48)$$

where c_{fi} denotes an interfacial friction coefficient, yields:

$$\left(-\frac{q_f^2}{g h_1^3} + \Phi\right) \frac{dh_1}{dx} = -\frac{q_f^2}{g h_1^3} c_{fi} \left(1 + (1 - \Phi) \frac{h_1}{h_2}\right) \quad (49)$$

which, dividing by Φ , can be rewritten as:

$$(1 - Fr_d^2) \frac{dh_1}{dx} = -Fr_d^2 c_{fi} \left(\frac{h_1 + h_2 - \Phi h_1}{h_2}\right) \approx -Fr_d^2 c_{fi} \frac{H}{h_2} \quad (50)$$

where Fr_d denotes the flow densimetric Froude number, defined as:

$$Fr_d^2 = \frac{q_f^2}{g \Phi h_1^3} \quad (51)$$

Equation (50) can be further reduced to:

$$\frac{dh_1}{dx} = -\frac{c_{fi} Fr_d^2 H}{(1 - Fr_d^2) h_2} \quad (52)$$

which is equivalent or analogous to the equation for backwater curves in open channel hydraulics, and can be used to determine the variation in x of the thickness of the upper layer along the salt wedge.

On the other hand, (47) can also be reduced to:

$$\frac{dh_2}{dx} = -(1 - \Phi) \frac{dh_1}{dx} + (1 - \Phi) c_{fi} \frac{q_f^2}{g h_1^3} \frac{h_1}{h_2} \quad (53)$$

which, neglecting terms of order Φ and c_{fi} , leads to:

$$\frac{dh_2}{dx} = -\frac{dh_1}{dx} \quad (54)$$

or:

$$\frac{d(h_1 + h_2)}{dx} = \frac{dH}{dx} = 0 \quad (55)$$

equation that shows that the total flow depth, $H = h_1 + h_2$, remains constant along the salt wedge, such that: $H = H_0$, where H_0 denotes the value of H at the location $x = 0$.

3 Length and shape of the arrested salt wedge

The densimetric Froude number can be rewritten as:

$$Fr_d^2 = \frac{q_f^2}{g \Phi_2 h_1^3} = \frac{q_f^2}{g \Phi_2 H_0^3} \frac{H_0^3}{h_1^3} = Fr_0^2 \frac{1}{r^3} \quad (56)$$

where the dimensionless parameter $r = h_1/H_0$ has been introduced.

With this definition:

$$\frac{H}{h_2} = \frac{H_0}{H_0 - h_1} = \frac{1}{1 - r} \quad (57)$$

which, replaced in (52), yields:

$$\frac{dr}{dX} = - \frac{Fr_0^2 c_{fi}}{r^3 (1 - Fr_0^2/r^3) (1 - r)} = \frac{c_{fi} Fr_0^2}{(Fr_0^2 - r^3) (1 - r)} \quad (58)$$

where:

$$r = \frac{h_1}{H_0} \quad ; \quad X = \frac{x}{H_0} \quad ; \quad Fr_0^2 = \frac{q_f^2}{g \Phi_2 H_0^3} \quad (59)$$

Assuming that c_{fi} is a constant, (58) can be integrated along the salt wedge, starting from its tip or upstream end ($x = 0$, where $r = 1$) so as to obtain the shape of the salt wedge. This leads to the following dimensionless expression:

$$Fr_0^2 r - \frac{r^4}{4} - \frac{1}{2} Fr_0^2 r^2 + \frac{r^5}{5} - \frac{1}{2} Fr_0^2 + \frac{1}{20} = c_{fi} Fr_0^2 X \quad (60)$$

The downstream boundary condition, at the outlet to the ocean or river mouth ($x = L$), corresponds to a critical flow condition, analogous to that of a final overfall in open channel flow. This critical flow condition is expressed, for the two-layer stratified flow in analysis, in terms of the densimetric Froude number of the flow, as has been discussed in a separate set of notes on internal hydraulics. This critical flow condition is thus expressed as:

$$Fr_d^2 = 1 = \frac{Fr_0^2}{r_c^3} \quad (61)$$

which yields:

$$r_c|_{X=L/H_0} = Fr_0^{2/3} \quad (62)$$

Replacing this condition in (60) leads finally to:

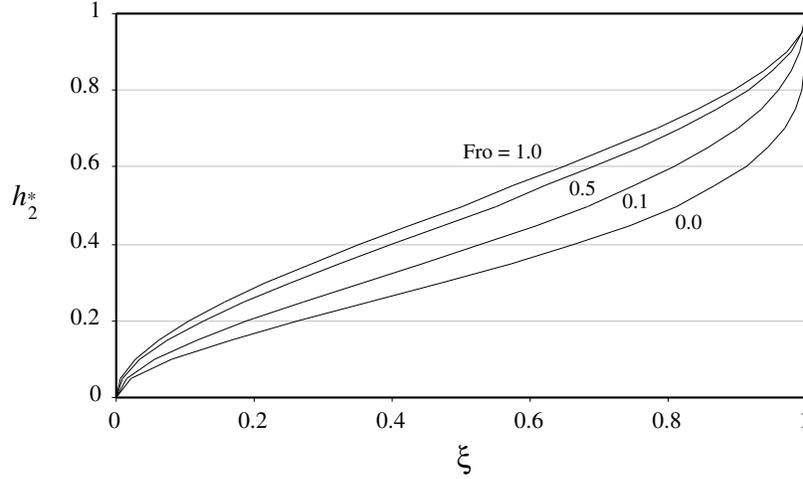


Figure 4: Dimensionless shape of salt wedge as a function of the densimetric Froude number Fr_0 .

$$\frac{L}{H_0} = \frac{1}{4 c f_i} \left\{ \frac{1}{5 Fr_0^2} - 2 + 3 Fr_0^{2/3} - \frac{6}{5} Fr_0^{4/3} \right\} \quad (63)$$

expression that can be used to estimate the length of intrusion of the arrested salt wedge, given the uniform fresh water flow depth, H , and corresponding volumetric discharge per unit width, q_f , and the Φ value of the salt water within the salt wedge. This solution for L was originally given by Schijf and Schonfeld in 1953.

Now, taking the ratio between (58) and (63), and defining the dimensionless variables:

$$\xi = \frac{x}{L} \quad ; \quad h_2^* = \frac{h_2}{h_{2c}} \quad ; \quad h_{2c} = H_0 (1 - r_c) \quad (64)$$

the following equation for the dimensionless shape of the salt wedge is obtained, which results to be just a function of Fr_0 :

$$(Fr_0^2 - 1) \frac{h_2^{*2}}{2} + (1 - Fr_0^{2/3}) h_2^{*3} \left\{ 1 - \frac{3}{4} (1 - Fr_0^{2/3}) h_2^* + \frac{1}{5} (1 - Fr_0^{2/3})^2 h_2^{*2} \right\} = K \xi \quad (65)$$

where:

$$K = -\frac{Fr_0^2}{4} \left(\frac{1}{5 Fr_0^2} - 2 + 3 Fr_0^{2/3} - \frac{6}{5} Fr_0^{4/3} \right) (1 - Fr_0^{2/3})^{-2} \quad (66)$$

The dimensionless shape of the salt wedge predicted by (65) is plotted in Fig. 4 for different values of Fr_0 .

4 References

- Fisher, H.B., List, E.J., Koh, R.C.Y, Imberger, J., and Brooks, N. (1979). Mixing in inland and coastal waters. Academic Press.

- Martin, J.L. and McCutcheon, S.C. (1999). Hydrodynamics and transport for water quality modelling. Lewis Publishers.
- Schijf, J.B. and Schonfeld, J.C. (1953). Theoretical considerations on the motion of salt and fresh water. Proceedings Minnesota International Hydraulics Convention, Minneapolis, Minnesota, pp. 321-333.
- Turner, J. S. (1973). Buoyancy Effects in Fluids. Cambridge University Press.