

# INTERNAL HYDRAULICS

CI 71Q HIDRODINAMICA AMBIENTAL    Profs. Y. Niño & A. Tamburrino  
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The flow of a fluid underneath a layer of stagnant, slightly lighter fluid, has properties that are common to those of open channel flow. In this set of lecture notes, a summary of the main results of what is known as *hydraulics of stratified flows* or simply *internal hydraulics* is presented. The idea is to analyze the behavior of the irrotational flow of a fluid with higher density than that of an overlying stagnant fluid, assuming that no mixing between them occurs. This no-mixing situation is possible in nature as long as the gradient Richardson number of the flow is large enough to preclude such mixing.

First, integral versions of the Energy (Bernoulli's) and Momentum equations are applied to the stratified flow in analysis. Afterwards, a number of typical examples of this type of flow are reviewed.

## 1 Bernoulli's Equation in Stratified Flows

Consider the situation shown in Fig. 1, where an irrotational flow, of thickness  $h$  and velocity  $u$ , of a fluid of density  $\rho_2$ , occurs below a layer of a different, stagnant fluid, of density  $\rho_1 < \rho_2$ . Assume that the upper layer has a thickness  $H \gg h$ .

The Navier-Stokes equation reads:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla \hat{p} + \nu \nabla^2 \vec{v} \quad (1)$$

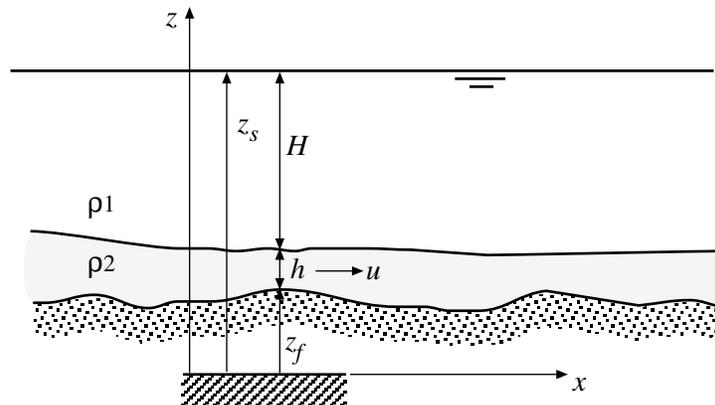


Figure 1: Flow of a heavier fluid below a layer of lighter stagnant fluid.

where  $\vec{v}$  denotes the velocity vector,  $\rho$  denotes the local density,  $\hat{p}$  denotes piezometric pressure and  $\nu$  denotes the local kinematic viscosity. Imposing the condition of irrotational flow:  $\vec{\omega} = \nabla \times \vec{v} = 0$ , the previous equation reduces to the well known Euler's equation:

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{2} \nabla(\vec{v} \cdot \vec{v}) = -\frac{1}{\rho} \nabla \hat{p} \quad (2)$$

In the case of a steady flow of a homogeneous fluid with velocity only in the  $x$  direction, such that:  $\vec{v} = u \hat{i}$ , Euler's equations reduces to:

$$\nabla \left( \frac{u^2}{2} + \frac{\hat{p}}{\rho} \right) = 0 \quad (3)$$

from where:

$$\frac{u^2}{2g} + \frac{\hat{p}}{\rho g} = B = \text{constant} \quad (4)$$

This is, of course, the well known Bernoulli's equation, showing that for irrotational flow, the Bernoulli,  $B$ , is conserved, that is, it remains constant within the whole flow field. In the context of the two-layer stratified flow in analysis, this equation applies to each layer, separately, where the density  $\rho$  is constant.

For the flow situation of Fig. 1, this equation yields:

$$B = \frac{u^2}{2g} + \frac{\hat{p}}{\rho_2 g} = C \quad (5)$$

where  $C$  represents a constant.

From Euler's equation in direction  $z$ , the hydrostatic pressure law, valid in each layer, is obtained such that:

$$\frac{\partial \hat{p}}{\partial z} = 0 \quad (6)$$

Call  $p_0$  the thermodynamic pressure on the free surface of the upper layer, located at a distance  $H + h$  from the bottom. The piezometric pressure at the density interface between upper and bottom layers is then given by:

$$\hat{p}_i = p_i + \rho_2 g z_i = p_0 + \rho_1 g H + \rho_2 g (z_f + h) \quad (7)$$

where  $p_i$  denotes the thermodynamic pressure at the interface and  $z_i$  and  $z_f$  denote the elevation  $z$  of the density interface and the bottom, respectively.

Since the surface level remains constant, then  $z_s = z_f + h + H = \text{constant}$ , therefore  $H = z_s - z_f - h$ . With this result, the piezometric pressure in the bottom layer reduces to:

$$\hat{p} = \hat{p}_i = p_0 + \rho_1 g z_s + (\rho_2 - \rho_1) g (z_f + h) \quad (8)$$

which, when introduced in Bernoulli's equation, yields:

$$B = \frac{u^2}{2g} + \frac{1}{\rho_2 g} (p_0 + \rho_1 g z_s + (\rho_2 - \rho_1) g (z_f + h)) = C \quad (9)$$

or:

$$B' = \frac{u^2}{2g} + \frac{(\rho_2 - \rho_1)}{\rho_2} (z_f + h) = C - \frac{p_0}{\rho_2 g} - \frac{\rho_1}{\rho_2} z_s \quad (10)$$

Calling  $\Delta\rho = \rho_2 - \rho_1$  and dividing the previous equation by  $\Delta\rho/\rho_2$  gives:

$$B'' = \frac{u^2}{2g'} + h + z_f = \frac{\rho_2}{\Delta\rho} \left( C - \frac{p_0}{\rho_2 g} - \frac{\rho_1}{\rho_2} z_s \right) = C' \quad (11)$$

where  $C'$  is a constant in this problem and  $g' = g \Delta\rho/\rho_2$ . This last term has been termed *reduced gravity* in previous lecture notes. The variable  $B'' = B \rho_2/\Delta\rho$  corresponds to the internal Bernoulli, which remains constant for the stratified flow in analysis. Hence, Bernoulli's equation for the internal flow is reduced to:

$$B'' = \frac{u^2}{2g'} + h + z_f = \text{constant} \quad (12)$$

expression that is identical to the classic Bernoulli's equation, except for the reduced gravity term.

The internal specific energy of the underflow is defined as  $B'' - z_f$ :

$$E = \frac{u^2}{2g'} + h \quad (13)$$

Consider flow in a constant width channel. The volumetric discharge per unit of fluid of density  $\rho_2$  is given by  $q = u h$ . The continuity equation applied to this flow is expressed simply as:

$$q = u h = \text{constant} \quad (14)$$

Replacing this result in the equation for  $E$  yields:

$$E = \frac{q^2}{2g' h^2} + h \quad (15)$$

expression that is identical to the specific energy equation of open channel hydraulics, except for the reduced gravity term. Just as in open channel hydraulics, it can be shown, by taking the derivative of equation (15) with respect to  $h$  and equating the result to zero, that the function  $E(h)$  has a minimum, for a given volumetric discharge per unit width,  $q$ , that is given by the condition:

$$Fr_d^2 = \frac{q^2}{g' h^3} = \frac{u^2}{g' h} = 1 \quad (16)$$

The term  $Fr_d$  is called *densimetric Froude number*. The condition  $Fr_d = 1$  corresponds to the so called *critical internal flow*. This critical flow occurs when the flow velocity  $u$ , equals the term  $\sqrt{g' h}$ . Recalling that for a two-layer stratified flow the celerity of internal waves is given by:

$$c_i = \sqrt{g' \frac{hH}{h+H}} \quad (17)$$

or, considering, just as it has been already assumed, that  $H \gg h$ :

$$c_i = \sqrt{g' h} \quad (18)$$

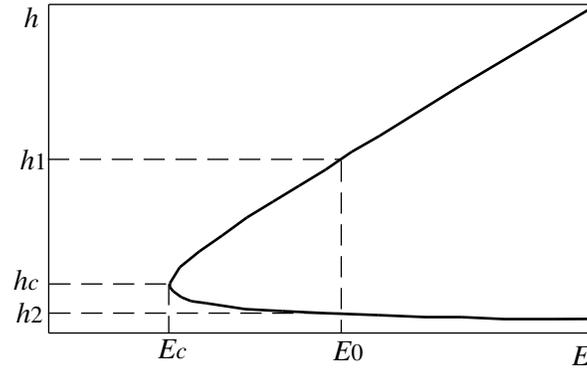


Figure 2: Internal specific energy as a function of the internal depth for a given volumetric discharge per unit width,  $q$ .

then it is concluded that the condition for internal critical flow corresponds to that for which the velocity of the internal flow equals the celerity of internal waves. Flows with velocities  $u < c_i$  are called *internal subcritical flows* and in them internal waves can travel upstream, against the current. Flows with velocities  $u > c_i$  are called *internal supercritical flows* and in them internal waves cannot travel upstream.

This is related to hydraulic controls of the internal flow. Internal subcritical flows are controlled by downstream flow conditions, internal supercritical flows are controlled only by upstream flow conditions.

The shape of the internal specific energy is shown in Fig. 2 for a given volumetric discharge per unit width,  $q$ . The minimum of  $E$  is denoted  $E_c$ , *minimum internal energy*, and corresponds to a flow depth,  $h_c$ , called *internal critical depth*. This is determined from (16) as:

$$h_c = \left(\frac{q^2}{g'}\right)^{1/3} \quad (19)$$

from where it is deduced that the minimum internal energy is given by:

$$E_c = \frac{3}{2} h_c \quad (20)$$

In order for an internal flow with a given volumetric discharge per unit width,  $q$ , to exist, the internal specific energy of the flow must be always larger than  $E_c$ . Fig. 2 shows that for fixed values of  $q$  and  $E_0 > E_c$  two different flow depths are possible, one greater than the internal critical depth,  $h_1 > h_c$ , thus corresponding to a subcritical internal flow, and the other lower than the internal critical depth,  $h_2 < h_c$ , corresponding to a supercritical internal flow. Which one of these two possible solutions prevails in a particular flow situation depends exclusively on the hydraulic controls acting upon the internal flow. This is further discussed below.

## 2 Momentum Equation in Stratified Flows

Consider the integral version of the momentum equation applied to the stratified flow in analysis. The situation shown in Fig. 3, where the internal flow faces an obstacle located on the bottom, is

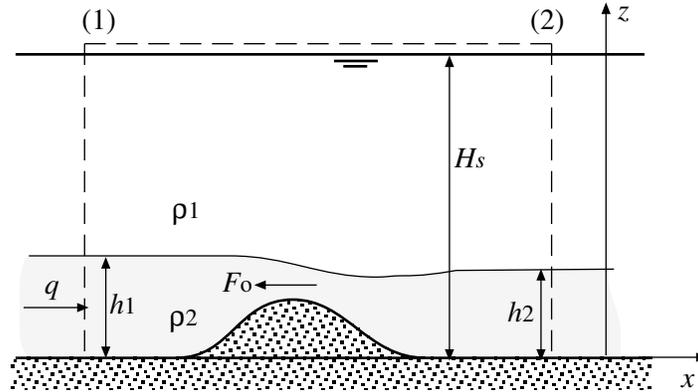


Figure 3: Control volume for the momentum equation applied to the analysis of internal flow over a bottom obstacle.

studied next. Just as in the previous analysis, the upper layer corresponds to stagnant water and it is considered to be so deep that its free surface is not deformed as a result of height variations of the internal flow. Call  $H_s$  the free surface elevation measured from the origin of the vertical axis  $z$ .

The momentum equation applied upon the control volume of Fig. 3, in the  $x$  direction, is expressed as:

$$\sum F_x = \rho_2 q (u_2 - u_1) \quad (21)$$

where  $q$  is the volumetric discharge per unit width of the fluid with density  $\rho_2$  flowing between sections (1) and (2), and  $u_1$  and  $u_2$  denote the inlet and outlet velocities in the  $x$  direction, respectively. The sum of the external forces per unit width in the  $x$  direction acting upon the control volume in this case are given by:

$$\sum F_x = Fp_1 - Fp_2 - F_0 \quad (22)$$

where  $Fp_1$  and  $Fp_2$  represent pressure forces per unit width acting on sections (1) and (2) of the control volume, respectively, and  $F_0$  denotes the force per unit width that the obstacle exerts upon the control volume, as a reaction to the drag force that the internal flow exerts upon it. In Eq. (22) the projection of the weight of the control volume in  $x$  direction has been neglected, assuming that the bottom is horizontal or of a rather small slope. The friction force between (1) and (2) can also be neglected, or else considered as included within the force  $F_0$ .

Assuming, as before, that the thermodynamic pressure at the free surface of the upper layer is  $p_0$  and that the hydrostatic law applies in both layers, then the pressure force in sections (1) and (2) of the control volume are given by:

$$Fp_1 = p_0 H_s + \rho_1 g \frac{(H_s - h_1)^2}{2} + \rho_1 g (H_s - h_1)h_1 + \rho_2 g \frac{h_1^2}{2} \quad (23)$$

$$Fp_2 = p_0 H_s + \rho_1 g \frac{(H_s - h_2)^2}{2} + \rho_1 g (H_s - h_2)h_2 + \rho_2 g \frac{h_2^2}{2} \quad (24)$$

respectively. Hence, the difference between pressure forces results to be:

$$Fp_1 - Fp_2 = (\rho_2 - \rho_1) g \frac{h_1^2}{2} - (\rho_2 - \rho_1) g \frac{h_2^2}{2} \quad (25)$$

Replacing this result in (22) and then in (21) yields:

$$(\rho_2 - \rho_1) g \frac{h_1^2}{2} - (\rho_2 - \rho_1) g \frac{h_2^2}{2} - F_0 = \rho_2 q(u_2 - u_1) \quad (26)$$

but from continuity:

$$q = u_1 h_1 = u_2 h_2 \quad (27)$$

so, replacing in (26) finally yields:

$$\frac{(\rho_2 - \rho_1)}{\rho_2} g \frac{h_1^2}{2} + u_1^2 h_1 = \frac{(\rho_2 - \rho_1)}{\rho_2} g \frac{h_2^2}{2} + u_2^2 h_2 + \frac{F_0}{\rho_2} \quad (28)$$

or:

$$\frac{h_1^2}{2} + \frac{q^2}{g' h_1} = \frac{h_2^2}{2} + \frac{q^2}{g' h_2} + \frac{F_0}{g' \rho_2} \quad (29)$$

Calling  $M$  the *internal specific force* per unit width or the *internal specific momentum function* per unit width, defined as:

$$M = \frac{h^2}{2} + \frac{q^2}{g' h} \quad (30)$$

then the momentum equation applied to the problem of Fig. 3 can be simply expressed as:

$$M_1 = M_2 + \frac{F_0}{g' \rho_2} \quad (31)$$

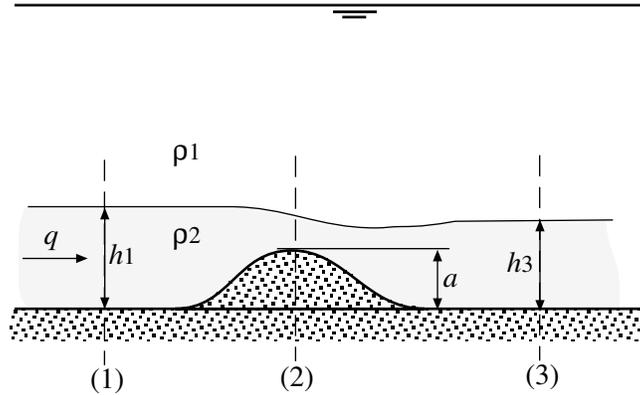
This equation is identical to that for the specific force of open channel hydraulics, except for the specific gravity term.

It can be shown, just as it was done for the internal specific energy, that the internal specific force function has a minimum, for a given volumetric discharge per unit width,  $q$ , which corresponds to the condition of internal critical flow. It is easy to show that the minimum of the internal specific force is given by:

$$M_c = \frac{3}{2} h_c^2 \quad (32)$$

with the critical depth  $h_c$  given by (19).

Likewise, for given values of the volumetric discharge per unit width,  $q$ , and the internal specific force,  $M_0 > M_c$ , two different internal flow depths are possible, one that corresponds to subcritical flow and the other that corresponds to supercritical flow. Again, the flow depth that prevails in a particular flow situation is determined by the hydraulic controls of the internal flow.

Figure 4: Internal flow over a sill of height  $a$ .

### 3 Application of the Internal Bernoulli's Equation

Consider an internal flow over a sill of height  $a$ , with a volumetric flow discharge  $q$  as shown in Fig. 4. Internal energy losses are neglected in the following analysis.

Assume that the internal flow downstream from the sill is subcritical so the flow depth in section (3),  $h_3$ , is controlled by flow conditions downstream. In this case, if the internal specific energy in (3) is large enough such that  $E_3 > E_c + a$ , then subcritical flow exists over the sill imposed by downstream conditions, such that  $E_2 = E_3 - a$ . The internal flow depth in section (2) selected from the two possible solutions that satisfy this equation, is that corresponding to subcritical flow:  $h_2 > h_c$ . The internal flow depth in (1) is obtained from  $E_1 = E_2 + a = E_3$ , where, again, the solution selected corresponds to subcritical flow, from where it is concluded that  $h_1 = h_3 > h_2 > h_c$  (Fig. 5). If, on the contrary, flow conditions are such that  $E_3 < E_c + a$ , then the flow in (3) does not have enough internal energy to control the flow in section (2). In this case internal critical flow is generated over the sill,  $h_2 = h_c$ , and the flow in section (2) is not controlled by flow conditions in section (3). In this case, there must be an internal energy loss downstream from the sill that dissipates the excess internal energy  $E_c + a - E_3$ . The internal critical flow in (2) imposes a subcritical flow in (1), such that  $E_1 = E_c + a$ , and the flow depth selected, of the two possible solutions of this equation, is that corresponding to subcritical flow:  $h_1 > h_c$ . In this flow situation  $h_1 > h_3 > h_2 = h_c$  (Fig. 6).

Assume now that the problem is controlled by upstream conditions. Accordingly, a supercritical internal flow exists in section (1), such that  $h_1 < h_c$ . If the internal energy in (1) is large enough so  $E_1 > E_c + a$ , then a supercritical internal flow is imposed over the sill such that  $E_2 = E_1 - a$ . The flow depth in (2), of the two possible solutions of this equation, corresponds to supercritical flow:  $h_2 < h_c$ . The flow depth in (3) is obtained from  $E_3 = E_2 + a = E_1$ , selecting the supercritical solution, so that:  $h_1 = h_3 < h_2 < h_c$  (Fig. 7). If, on the contrary,  $E_1 < E_c + a$ , then the internal flow in (1) does not have enough energy to control the flow in (2). In this case critical flow is generated over the sill,  $h_2 = h_c$ , and in (1) a subcritical flow is imposed, with internal energy  $E'_1 = E_c + a > E_1$ . That is, the sill is a hydraulic control for the internal flow, increasing the flow depth upstream to  $h'_1 > h_c > h_1$ . Downstream, the internal flow results to be supercritical, with an internal energy level  $E_3 = E_c + a = E'_1$ . The corresponding flow depth selected is  $h_3 < h_c$  (Fig. 8).

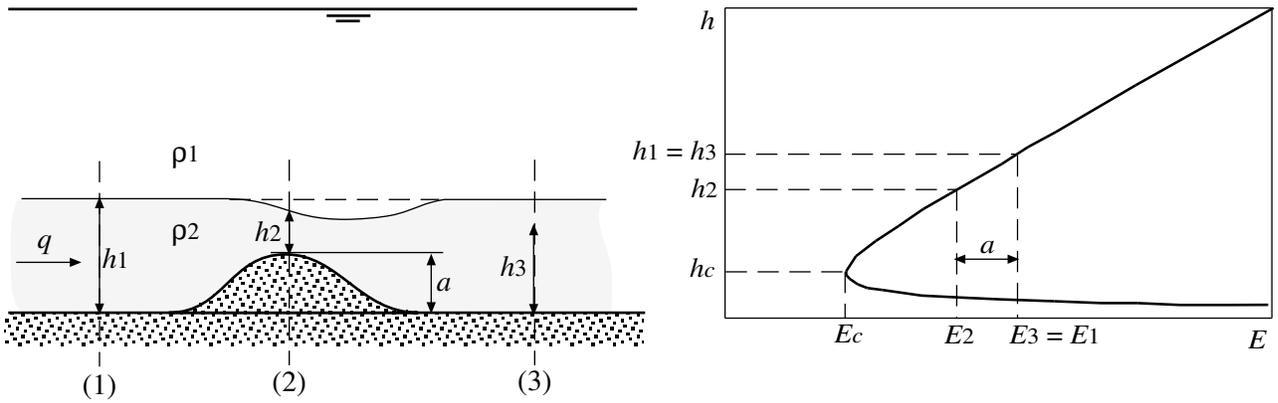


Figure 5: Subcritical internal flow over sill.

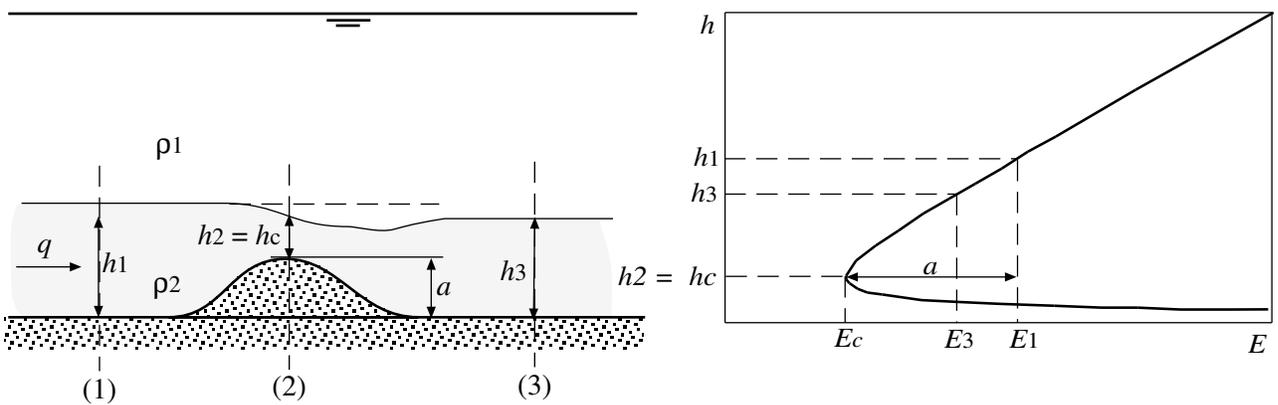


Figure 6: Internal critical flow over sill. Flow conditions in (3) do not control the flow in (2).

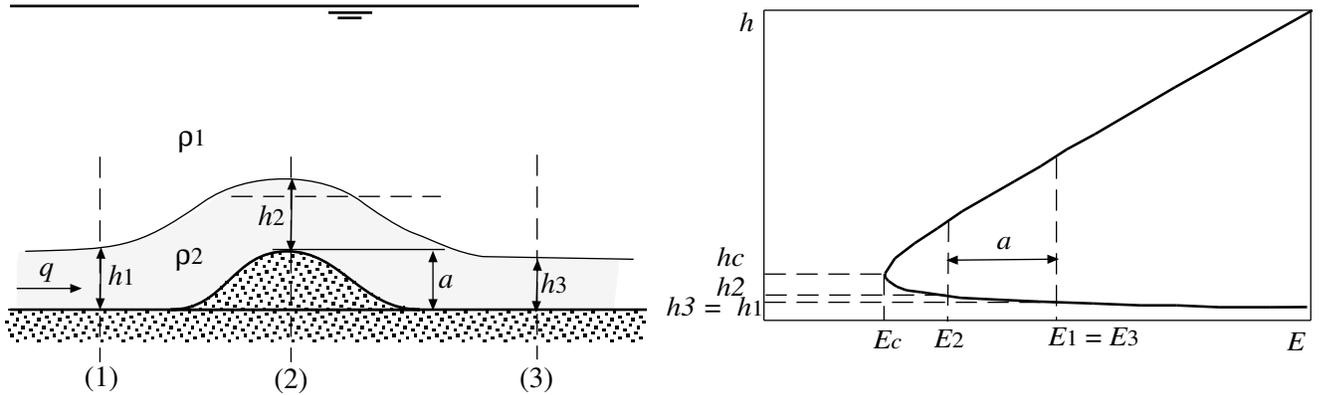


Figure 7: Supercritical internal flow over sill.

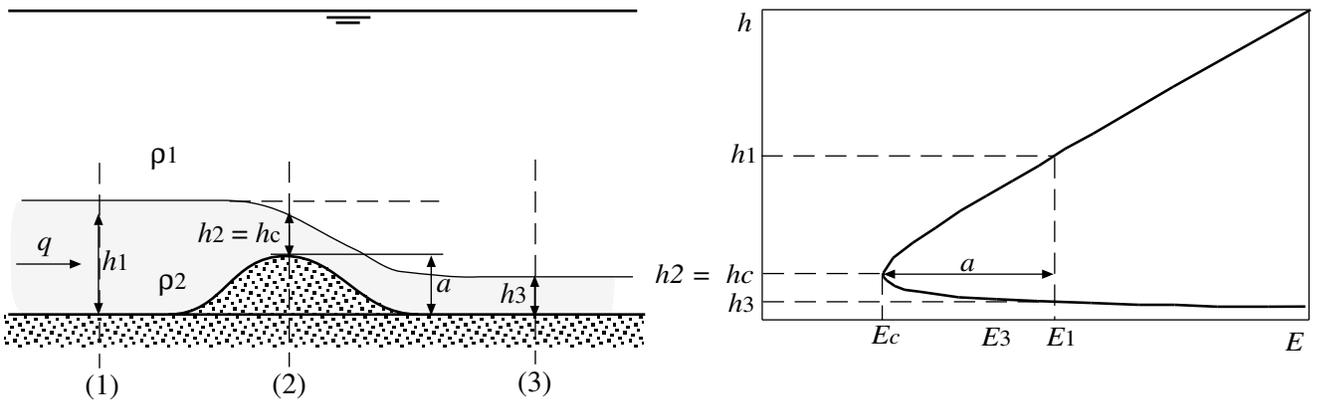


Figure 8: Internal critical flow over the sill. The internal flow in section (1) is subcritical and in section (3) is supercritical.

### 4 Internal Hydraulic Jump

Just as in the case of open channel hydraulics, whenever there is a conflict regarding hydraulic controls, such that a supercritical internal flow is imposed by upstream conditions and a subcritical internal flow is imposed by downstream conditions, the conflict is resolved by means of an *internal hydraulic jump*, which is equivalent to the classic hydraulic jump of open channel flows.

In the flow situation of Fig. 9, the internal specific force is conserved between sections (1) and (2). In fact, neglecting the friction force, Eq. (31) reads:

$$M_1 = \frac{h_1^2}{2} + \frac{q^2}{g' h_1} = M_2 = \frac{h_2^2}{2} + \frac{q^2}{g' h_2} \tag{33}$$

After some algebra this equation can be written in a form equivalent to that of the Belanger equations, relating the conjugate flow depths of the internal hydraulic jump.

$$\frac{h_2}{h_1} = \frac{1}{2} \left\{ \sqrt{1 + 8 Fr_{d1}^2} - 1 \right\} \tag{34}$$

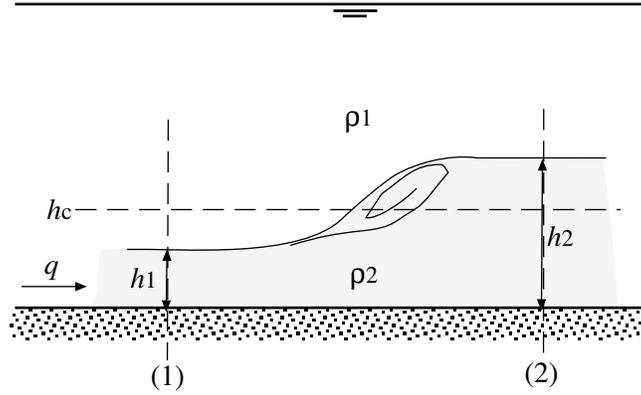


Figure 9: The internal hydraulic jump.

$$\frac{h_1}{h_2} = \frac{1}{2} \left\{ \sqrt{1 + 8 Fr_{d2}^2} - 1 \right\} \quad (35)$$

where  $Fr_{d1}^2 = q^2/(g' h_1^3)$  and  $Fr_{d2}^2 = q^2/(g' h_2^3)$  denote the upstream and downstream densimetric Froude numbers of the internal hydraulic jump, respectively.

From (33), the internal energy loss associated to the sudden flow expansion in the internal hydraulic jump can be estimated as:

$$\Lambda_j = E_1 - E_2 = \frac{(h_2 - h_1)^3}{4 h_1 h_2} \quad (36)$$

## 5 Internal Flow on an Inclined Plane

Consider a fluid of density  $\rho_2$ , flowing beneath a layer of a lighter, stagnant fluid of density  $\rho_1$ , along an inclined plane with an angle  $\theta$  with respect to the horizontal. The free surface of the upper layer can be considered to be horizontal with a constant pressure  $p_0$ . Applying the integral version of the momentum equation to a small control volume, such as that shown in Fig. 10, the following result is easily obtained:

$$F_p + W \sin \theta - F_\tau = \rho_2 \frac{d(u^2 h)}{dx} dx \quad (37)$$

where  $F_p$  denotes the net pressure force acting upon the control volume,  $W$  denote the weight of the control volume,  $F_\tau$  denotes the total friction force acting on the control volume, and  $u$  and  $h$  correspond to the velocity and local depth of the internal flow, respectively.

The total pressure force acting on the control volume is:

$$F_p = -(\rho_1 g (z_s - z_f) \tan \theta + (\rho_2 - \rho_1) g h \frac{dh}{dx} \cos \theta) dx \quad (38)$$

where  $z_s$  and  $z_f$  denote the local free surface and bottom elevations, respectively. The weight of the control volume is given by:

$$W = \left( \frac{\rho_1 g (z_s - z_f)}{\cos \theta} + (\rho_2 - \rho_1) g h \right) dx \quad (39)$$

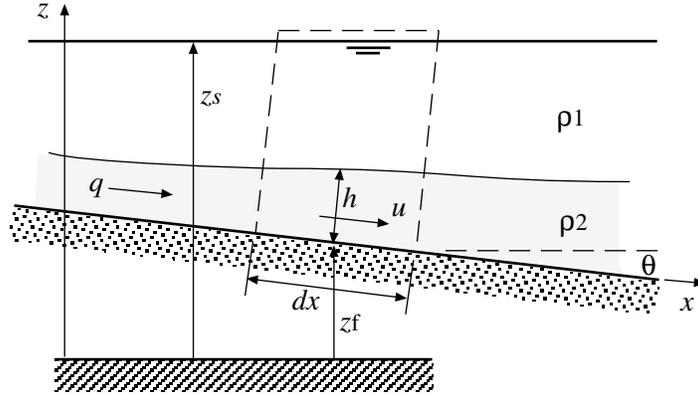


Figure 10: Gradually varied internal flow on a plane inclined in an angle  $\theta$  with respect to the horizontal.

On the other hand, the total friction force acting on the control volume is given only by the shear stress exerted by the bottom wall on the internal flow. If  $\tau_b$  denote the bottom shear stresses, then introducing a friction factor,  $C_f$ , it can be modeled as:

$$\tau_b = \rho_2 C_f u^2 \quad (40)$$

and the total friction force acting on the control volume is given by:

$$F_\tau = \rho_2 C_f u^2 dx \quad (41)$$

With these considerations, the differential equation modeling the internal flow over the inclined plane is written as:

$$\rho_2 \frac{d(u^2 h)}{dx} = (\rho_2 - \rho_1) g h \sin \theta - (\rho_2 - \rho_1) g h \frac{dh}{dx} \cos \theta - \rho_2 C_f u^2 \quad (42)$$

or, for angles  $\theta$  lower than about  $10^\circ$ , such that  $\cos \theta \approx 1$ :

$$\frac{d(u^2 h)}{dx} = g' h \sin \theta - g' h \frac{dh}{dx} - C_f u^2 \quad (43)$$

This equation can be rewritten using the continuity equation:  $q = u h = \text{constant}$ . Indeed, since:

$$\frac{dq}{dx} = \frac{d(u h)}{dx} = h \frac{du}{dx} + u \frac{dh}{dx} = 0 \quad (44)$$

then,

$$\frac{d(u^2 h)}{dx} = h \frac{du^2}{dx} + u^2 \frac{dh}{dx} = h \frac{du^2}{dx} - u h \frac{du}{dx} = \frac{1}{2} h \frac{du^2}{dx} \quad (45)$$

Replacing this equation in (43), yields:

$$\frac{1}{2 g'} \frac{du^2}{dx} - \sin \theta + \frac{dh}{dx} = -\frac{C_f u^2}{g' h} \quad (46)$$

or, since  $\sin \theta = -dz_f/dx$ :

$$\frac{d}{dx} \left( \frac{u^2}{2g'} + z_f + h \right) = -\frac{C_f u^2}{g' h} \quad (47)$$

The longitudinal gradient of the internal Bernoulli appears in the left hand side of the previous equation, so:

$$\frac{dB''}{dx} = \frac{\rho_2}{\Delta \rho} \frac{dB}{dx} = -\frac{C_f u^2}{g' h} \quad (48)$$

Since  $dB/dx = -S_f$ , where  $S_f$  represents the frictional loss of energy per unit length, or simply the friction slope, then it is concluded that:

$$S_f = \frac{\tau_b}{\rho_2 g h} = \frac{C_f u^2}{g h} \quad (49)$$

a result that is analogous to that of open channel hydraulics.

Going back to Eqn. (46) and calling  $S = \sin \theta$  the bottom slope, yields:

$$\frac{1}{2 g'} \frac{du^2}{dx} - S + \frac{dh}{dx} = -\frac{C_f u^2}{g' h} \quad (50)$$

Replacing now the result  $u = q/h$  in the previous equation gives:

$$\frac{1}{2 g'} \frac{d}{dx} \left( \frac{q^2}{h^2} \right) - S + \frac{dh}{dx} = -\frac{C_f q^2}{g' h^3} \quad (51)$$

which finally yields:

$$-\frac{q^2}{g' h^3} \frac{dh}{dx} - S + \frac{dh}{dx} = -\frac{C_f q^2}{g' h^3} \quad (52)$$

Since the densimetric Froude number has been defined as:  $Fr_d^2 = q^2/(g' h^3)$  then this equation reduces to:

$$(1 - Fr_d^2) \frac{dh}{dx} = S - C_f Fr_d^2 \quad (53)$$

which can be used to determine the variation of the internal flow depth along the inclined plane, as a function of the bottom slope,  $S$ , the volumetric discharge per unit width,  $q$ , and the bottom friction factor,  $C_f$ . It is convenient to note that the densimetric Froude number changes with the flow depth and therefore it is not a constant of the problem. This equation is analogous to that for gradually varied flow in open channels.

From this equation it is concluded that in this non-mixing internal flow, the existence of a uniform underflow is possible, for which the depth remains constant along the inclined plane. Actually, such flow corresponds to an equilibrium situation in which the longitudinal pressure gradient vanishes and the reduced gravity is balanced by the friction force acting upon the internal flow. The resulting uniform flow depth is called *internal normal depth*,  $h_n$ . Imposing the condition  $dh/dx = 0$  in (53), yields:

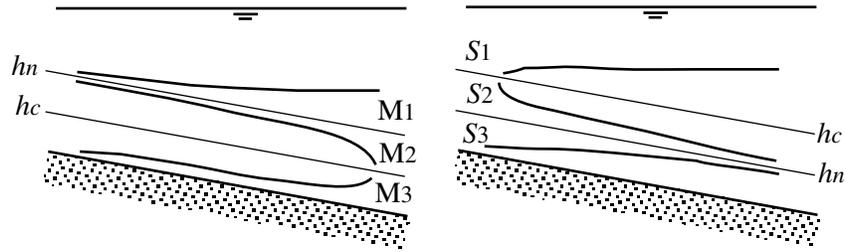


Figure 11: Classification of gradually varied internal flows for mild and steep slopes.

$$S = S_f = C_f \frac{q^2}{g' h_n^3} \quad (54)$$

from which an expression for the internal normal depth is obtained:

$$h_n = \left( \frac{C_f q^2}{g' S} \right)^{1/3} \quad (55)$$

Depending on the values of the internal critical and normal depths, the internal flows can be classified as occurring in *mild* or *steep slopes*. A mild slope internal flow occurs when  $h_n > h_c$  (that is, when the normal internal flow is subcritical); a steep slope internal flow occurs when  $h_n < h_c$  (that is, when the normal internal flow is supercritical). Given this classification and the equation for the gradually varied internal flow (53), it is possible to deduce the different internal flow situations that can possibly occur. They are represented in Fig. 11, in analogy with open channel hydraulics.

Changes in bottom slope can generate changes of internal hydraulic slope. Two typical examples are shown in Fig. 12. In the first case, the change in bottom slope  $S_2 > S_1$  generates a change in internal hydraulic slope, such that the upstream reach is of mild slope and the downstream reach is of steep slope. In this situation the hydraulic control occurs at the point of bottom slope change. Critical internal flow occurs there and a type M2 subcritical internal flow is generated in the upstream reach and a type S2 supercritical internal flow is generated in the downstream reach. The normal internal flow depths,  $h_{n1}$  and  $h_{n2}$  are reached asymptotically upstream and downstream, respectively. In the second example of Fig. 12:  $S_1 > S_2$ , which results in a steep slope for the upstream reach and a mild slope for the downstream reach. The internal flow depth upstream is supercritical and that downstream is subcritical. The compatibility of the conflicting hydraulic controls is attained through an internal hydraulic jump.

## 6 The Lock Exchange Problem

Consider now the problem of a fluid of density  $\rho_2$  intruding below a fluid of lower density  $\rho_1$ . In this case the analysis of previous sections is valid for the internal steady flow generated upstream from the intruding front. In this section the motion of the front is analyzed. Examples of this situation can be, for instance, the motion of cold air underneath warm air in the atmosphere, or the intrusion of saline water underneath fresh water, when a sluice gate separating both liquids is opened. This latter case is the one that gives its name to this classic problem: “lock exchange”.

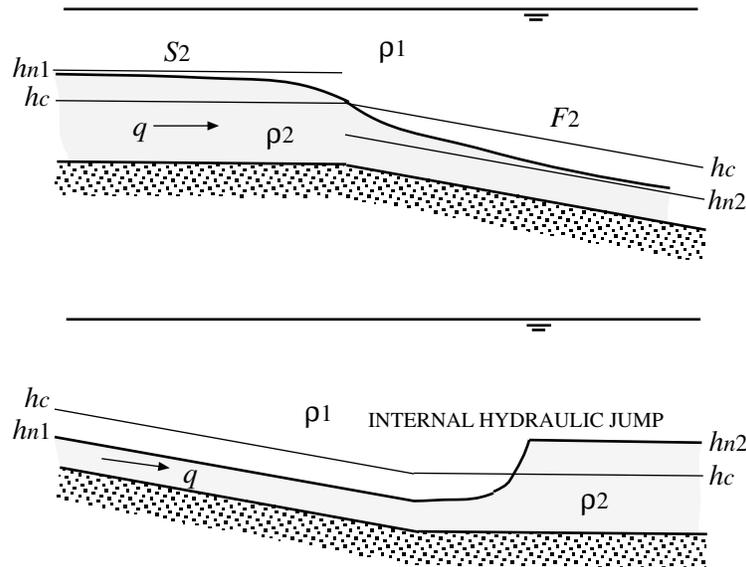


Figure 12: Changes in internal hydraulic slope. Case 1: mild to steep slope. Case 2: steep to mild slope.

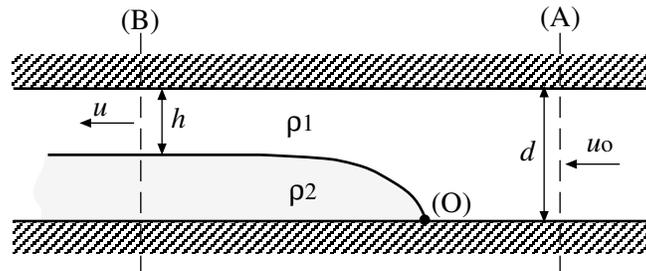


Figure 13: Displacement of a fluid front in a close conduit filled with another fluid of lesser density. The system of reference is attached to the front.

The case of a frictionless two-dimensional flow in a closed horizontal channel of height  $d$ , such as that shown in Fig. 13, is analyzed first. Considering a system of reference attached to the moving front that has a constant velocity  $u_0$ , it is easy to see that the problem is equivalent to that of the flow of a fluid of density  $\rho_1$  directed, with a velocity  $u_0$ , towards the static front of density  $\rho_2$ . The less dense fluid moves over the static heavier fluid, reaching a thickness  $h$  and a velocity  $u$  above the front, just as is shown in Fig. 13.

Applying Bernoulli's equation between sections (A) and (B) of Fig. 13, assuming that the hydrostatic pressure law is valid and neglecting energy losses, gives:

$$\frac{u_0^2}{2g} + \frac{p_A}{\rho_1 g} + d = \frac{u^2}{2g} + \frac{p_B}{\rho_1 g} + d \tag{56}$$

where  $p_A$  and  $p_B$  denote the pressure at the top wall of the conduit in sections (A) and (B), respectively. Since the front of fluid of density  $\rho_2$  remains still with respect to the system of reference, then point (O) of Fig. 13 is a stagnation point. This implies that the following pressure

balance exists in (0):

$$p_0 = p_A + \rho_1 g d + \rho_1 g \frac{u_0^2}{2g} = p_B + \rho_1 g h + \rho_2 g (d - h) \quad (57)$$

from where it is concluded that:

$$\frac{p_A}{\rho_1 g} - \frac{p_B}{\rho_1 g} = \frac{(\rho_2 - \rho_1)}{\rho_1} (d - h) - \frac{u_0^2}{2g} \quad (58)$$

Replacing this equation in (57) yields:

$$u^2 = 2 g' (d - h) \quad (59)$$

On the other hand, applying the integral version of momentum equation to the control volume including sections (A) and (B) yields:

$$Fp = \rho_1 (u^2 h - u_0^2 d) \quad (60)$$

where  $Fp$  is the resulting pressure force acting upon the control volume, which is given by the difference between the pressure forces acting on sections (A) and (B):

$$Fp = p_A d + \rho_1 g \frac{d^2}{2} - p_B d - \rho_1 g \frac{h^2}{2} - \rho_1 g h (d - h) - \rho_2 g \frac{(d - h)^2}{2} \quad (61)$$

or, simplifying the above equation:

$$Fp = \frac{1}{2} (\rho_2 - \rho_1) g (d^2 - h^2) - \rho_1 \frac{u_0^2 d}{2} \quad (62)$$

Replacing this result in (60) yields:

$$\frac{1}{2} (u_0^2 d + g' d^2) = u^2 h + \frac{1}{2} g' h^2 \quad (63)$$

Considering now the continuity equation,  $q = u_0 d = u h$ , gives:

$$u^2 = g' \frac{d}{h} \frac{(d^2 - h^2)}{(2d - h)} \quad (64)$$

And equating (59) and (64):

$$2 g' (d - h) = g' \frac{d}{h} \frac{(d^2 - h^2)}{(2d - h)} \quad (65)$$

which has a non-trivial solution (i.e.,  $h \neq d$ ) given by:

$$h = \frac{d}{2} \quad (66)$$

that is, in this case the front fills half the total height of the conduit along which it is moving. Replacing this result in (59) or (64) yields:

$$u = \sqrt{g' d} \quad (67)$$

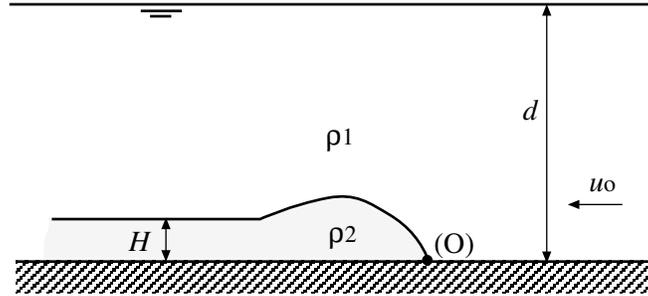


Figure 14: Displacement of a front of fluid underneath a very deep ambient layer of less dense stagnant fluid. The system of reference is attached to the front.

$$u_0 = \frac{1}{2} \sqrt{g' d} \quad (68)$$

This last equation predicts that the displacement velocity of the front of density  $\rho_2$  underneath the fluid of density  $\rho_1$  in the close conduit, is given by the dimensionless relationship:  $u_0/\sqrt{g' d} = 1/2$ . This same relationship shows that the front is subcritical with a value of the densimetric Froude number  $Fr_{d0} = 1/2$ . On the other hand, the flow on top of the front is supercritical, with  $u/\sqrt{g' h} = \sqrt{2}$ .

Experimental results show that, for sufficiently large Reynolds numbers, the front velocity in the flow situation of Fig. 13 is indeed constant, and is given by  $Fr_{d0} = u_0/\sqrt{g' d} = 0.44$ . This result is close enough to that obtained theoretically through the analysis presented previously. The differences with the theory are probably due to friction effects, which were neglected in the theoretical analysis. Friction with the bottom is said to be more important than that occurring at the density interface. In the case of an ambient fluid with a free surface, it has been found that the flow becomes asymmetric. The front velocity in this case is given by  $u_0/\sqrt{g' d} = 0.47$  if the front is submerged, or  $u_0/\sqrt{g' d} = 0.59$  if the front moves along the surface.

When the intruding front thickness is much smaller than the height of the ambient fluid, it has been observed that an energy loss exists in the vicinity of the front. Such energy loss occurs through the breaking of internal waves behind the front, related to Kelvin-Helmholtz instability of the density interface. Experimental observations have shown that density currents in deep waters have a *head* with a thickness that is about twice the thickness  $H$  of the uniform current following the front (Fig. 14). Behind the front a turbulent zone is produced together with an abrupt change in the internal current height. The drag force created by the associated change of internal flow velocity must be taken into account when estimating the force balance acting upon the front.

Neglecting the mixing that takes place in the wake of the front head and assuming that the pressure distribution within the wake is hydrostatic, then the velocity of the internal flow of thickness  $H$  shown in Fig. 14,  $u_0$ , can be estimated as follows. Considering that the dynamic pressure in the stagnation point (O) at the front head must be equal to the pressure difference along the bottom, at sections sufficiently distant from the front both in the upstream and downstream directions, then:

$$\rho_1 g (d - H) + \rho_2 g H = \rho_1 g \frac{u_0^2}{2g} + \rho_1 g d \quad (69)$$

from where it is deduced that:

$$u_0 = \sqrt{2} \sqrt{g' H} \quad (70)$$

Experimental observations, which include, of course, mixing effects in the wake of the front as well as friction effects (which tend to lift the stagnation point of the front from the bottom wall), give a value to the proportionality coefficient in (70) of 1.1, which is slightly lower than the theoretical value of  $\sqrt{2}$  just found.

Recalling that for the close conduit:  $u_0 = 0.5\sqrt{g' d}$  and as it was shown,  $d - h = H = d/2$ , then the velocity of the front inside the conduit, expressed as a function of the current thickness,  $H$ , is:  $u_0 = (\sqrt{2}/2) \sqrt{g' H}$ . Based on this result, it can be noted that if the thickness of the upper layer, or ambient fluid,  $d - H$ , is not large enough compared with  $H$ , then the proportionality factor in (70) vary between the value  $\sqrt{2}/2$ , obtained for the internal flow in the close conduit, to the value  $\sqrt{2}$ , corresponding to the case  $H/d \rightarrow 0$ . It has been determined experimentally that even in cases in which  $H/d \approx 0.1$  the correction of the coefficient of proportionality in (70) caused by a finite thickness of the ambient fluid layer can be significative.

Finally, experiments conducted in the case of a non horizontal bottom wall have shown that for a range of slopes between 0 and 1/20 the velocity of the front results to be a function of the head height,  $D$ , and the local density difference between the head and the ambient fluid, which is given by the empirical relationship:

$$u_0 = 0.75 \sqrt{g' D} \quad (71)$$

## 7 References

- Simpson, J.E. (1997). Gravity currents in the Environment and the Laboratory. Second Edition. Cambridge University Press.
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