PLUNGING FLOW INTO A RESERVOIR

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1 Inflow Mixing

Inflows contribute to the mixing of lakes and reservoirs and serve as a primary source of dissolved and particulate materials. The inflow rate and density difference between the inflowing water and the water body both dictate the characteristics of the inflow mixing processes.

The initial momentum of an inflow pushes the lake or reservoir more stagnant water ahead of it. This occurs along a distance until the initial inflow momentum is substantially dissipated by the bottom shear stress and the longitudinal pressure gradient created by the increasingly deeper waters of the water body. In this transitional region, the transport is dominated by advection and there is high enough turbulent kinetic energy levels to maintain the inflowing water well mixed. However, as the inflow losses its momentum, buoyancy associated with density differences between the inflowing water and the water body dominate the transport.

Two different situations may occur depending on whether the inflow water has a higher or lower density, $\rho_a + \Delta\rho_0$, than that of the surface layer of the lake or reservoir, ρ_a. If $\Delta\rho_0$ is negative, a surface density current or overflow is created that propagates along the surface of the water body, and the lighter inflow water separates from the bottom (Fig. 1). If $\Delta\rho_0$ is positive, plunging of the inflow occurs and a bottom density current or underflow occurs that propagates along the bottom of the water body (Fig. 2). If the water body is stratified, an interflow can occur (as shown in Fig. 1) whenever the underflow reaches a region of matching density. A bottom density current occurs as long as the density of the underflow remains larger than that of the bottom layer of the water body.

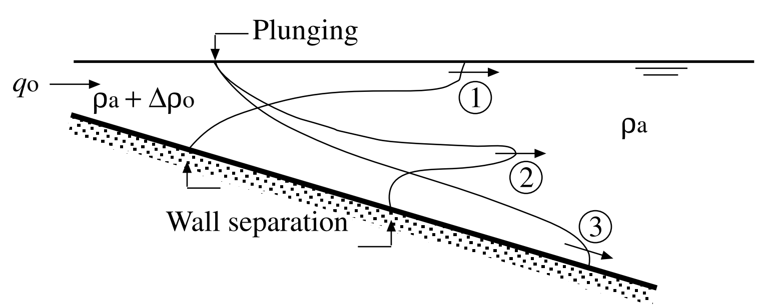

Figure 1: Inflow mixing. (1) surface density current or overflow; (2) interflow; (3) bottom density current or underflow.

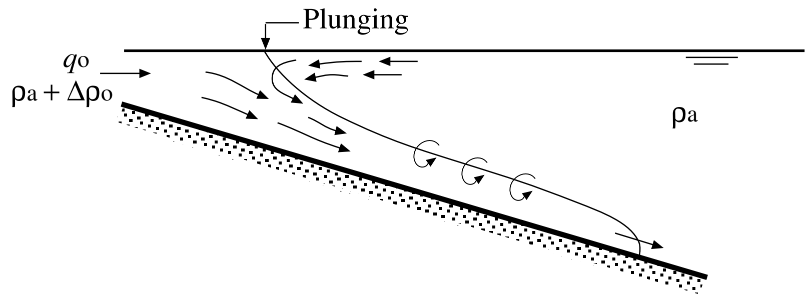

Figure 2: Plunging in heavy density currents. Mixing occurs at the plunge point and at the density interface of the underflow.

Inflow mixing occurs associated with the density currents created inside the water body. If the gradient Richardson number of the stratified flows is low enough then interface mixing will occur at the expense of the turbulent kinetic energy of the buoyancy driven flow, as discussed in previous lecture notes. However, another important mixing process may occur associated with underflows. In some cases, when the plunging of inflow water is rather energetic, it creates mixing at the plunge region, where ambient water from the lake or reservoir is entrained into the underflow, diluting the inflow density and increasing the volumetric discharge of the underflow (Fig. 2).

In what follows an analysis of underflows and the plunging phenomenon is presented, following the ideas of Akiyama and Stefan (1984).

2 Plunging Flow into a Reservoir

The geometry of the flow cross section is assumed to be rectangular and the ambient density, ρ_a, is assumed to be constant. That is, the water body is considered to be non-stratified. Consider an inflow discharge per unit width, q_0, with a density $\rho_a + \Delta\rho_0$, a depth h_0 and a velocity u_0. Applying conservation of volume to the control volume of Fig. 3, yields:

$$q_0 = u_0\, h_0 = u_p\, h_p = u_d\, h_d - u_a\, h_a \tag{1}$$

where h_p and u_p denote the depth and flow velocity at the plunge point, respectively, h_d and u_d denote the height and flow velocity of the underflow just downstream from the plunge point, respectively, and h_a and u_a denotes the height and flow velocity of the counter current of ambient water that will eventually mix with the inflowing water in the plunge region. It is assumed that the ambient flow discharge, q_a, is related to the inflow discharge, q_0, by a *mixing coefficient*, γ:

$$q_a = u_a\, h_a = \gamma\, q_0 \tag{2}$$

The underflow discharge is then:

$$q_d = u_d\, h_d = (1 + \gamma)\, q_0 \tag{3}$$

Applying conservation of mass to the same control volume of Fig. 3, assuming that the density of the underflow changes, due to plunging mixing, to $\rho_a + \Delta\rho_d$, yields:

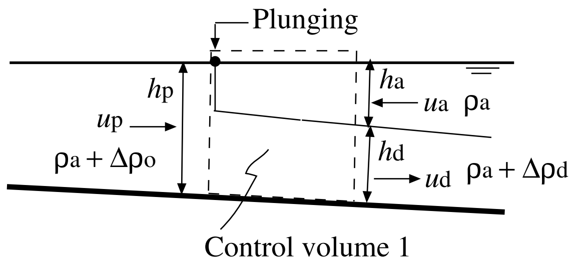

Figure 3: Control volume 1 for continuity and momentum equations.

$$(\rho_a + \Delta\rho_0)\; q_0 = (\rho_a + \Delta\rho_d)\; q_d - \rho_a\; q_a \tag{4}$$

which, combined with (1), (2) and (3), gives:

$$\Delta\rho_0 = (1+\gamma)\;\Delta\rho_d \tag{5}$$

or, defining $\epsilon_0 = \Delta\rho_0/\rho_a$ and $\epsilon_d = \Delta\rho_d/\rho_a$:

$$\frac{\epsilon_d}{\epsilon_0} = \frac{1}{1+\gamma} \tag{6}$$

Now, definitions for the densimetric Froude numbers of the inflow, F_0, plunging flow, F_p, and underflow, F_d, are introduced as:

$$F_0^2 = \frac{q_0^2}{\epsilon_0\; g\; h_0^3} \quad ; \quad F_p^2 = \frac{q_0^2}{\epsilon_0\; g\; h_p^3} \quad ; \quad F_d^2 = \frac{q_d^2}{\epsilon_d\; g\; h_d^3} \tag{7}$$

Using the definition $K = h_p/h_d$, a relationship between F_d and F_p readily follows as:

$$F_d^2 = (1+\gamma)^3\; K^3\; F_p^2 \tag{8}$$

Applying the momentum equation in the longitudinal direction to the control volume 1 shown in Fig. 3, using Boussinesq approximation to neglect density differences with respect to the ambient water in the inertia terms and assuming a rather small bottom slope so that the component of gravity in the longitudinal direction can be neglected and $h_a \approx h_p - h_d$, yields:

$$\rho_a\; u_d^2\; h_d - \rho_a\; u_p^2\; h_p + \rho_a\; u_a^2\; h_a = \frac{g}{2}\;(\rho_a + \Delta\rho_0)\; h_p^2 - \frac{g}{2}\;\rho_a\; h_a^2 - \rho_a\; g\; h_a\; h_d - \frac{g}{2}\;(\rho_a + \Delta\rho_d)\; h_d^2 \tag{9}$$

which can be simplified to:

$$u_d^2\; h_d - u_p^2\; h_p + u_a^2\;(h_p - h_d) = \frac{g}{2}\;(\epsilon_0\; h_p^2 - \epsilon_d\; h_d^2) \tag{10}$$

Now, applying the momentum equation in the longitudinal direction to the control volume 2 shown in Fig. 4, assuming that the plunge point is a stagnation point for the inflow, yields:

$$\rho_a\; u_a^2\; h_a = \frac{g}{2}\;(\rho_a + \Delta\rho_0)\; h_a^2 - \frac{g}{2}\;\rho_a\; h_a^2 \tag{11}$$

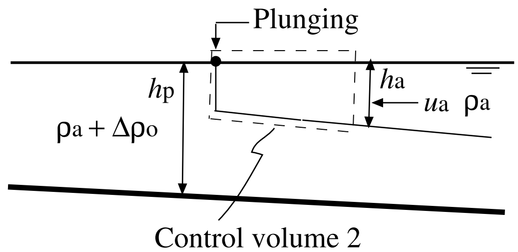

Figure 4: Control volume 2 for the momentum equation. It is assumed that the plunge point is a stagnation point for the inflow.

which can be simplified to:

$$u_a^2 = \frac{g}{2}\,\epsilon_0\,(h_p - h_d) \tag{12}$$

Combining (10) and (12) gives:

$$F_d^2 - \frac{F_d^2}{(1+\gamma)^2}\,\frac{h_d}{h_p} = (1+\gamma)\,\frac{h_p}{h_d} - \frac{2+\gamma}{2} \tag{13}$$

or introducing the parameter $K = h_p/h_d$ defined before:

$$F_d^2 - \frac{F_d^2}{(1+\gamma)^2}\,\frac{1}{K} = (1+\gamma)\,K - \frac{2+\gamma}{2} \tag{14}$$

which provides the following quadratic equation for K:

$$K^2 - \frac{1}{(1+\gamma)}\,(\frac{2+\gamma}{2} + F_d^2)\,K + \frac{F_d^2}{(1+\gamma)^3} = 0 \tag{15}$$

which has the solution:

$$K = \frac{1}{2\,(1+\gamma)}\,\{\frac{2+\gamma}{2} + F_d^2 + \sqrt{\left(\frac{2+\gamma}{2} + F_d^2\right)^2 - \frac{4\,F_d^2}{(1+\gamma)}}\,\} \tag{16}$$

where only the positive sign in the solution of (15) has been used, as it is the one that makes physical sense. It is easy to see that the condition:

$$\left(\frac{2+\gamma}{2} + F_d^2\right)^2 - \frac{4\,F_d^2}{(1+\gamma)} \geq 0 \tag{17}$$

is always met for $\gamma \geq 0$.

In order for the present analysis to be valid, $K \geq 1$ is required, which puts the following restriction on F_d^2:

$$F_d^2 \geq \frac{\gamma\,(1+\gamma)^3}{2\,\left((1+\gamma)^3 - (1+\gamma)\right)} \tag{18}$$

The solution obtained so far shows that the underflow conditions determine the plunge point conditions. Indeed, a value of the underflow densimetric Froude number, F_d, needs to be specified externally to the present model. This, from (8), determines the plunge point densimetric Froude number as:

$$F_p^2 = \frac{F_d^2}{(1+\gamma)^3 \; K^3} \tag{19}$$

and from (7) and (8) the dimensionless plunge point depth as:

$$\frac{h_p}{h_0} = \left(\frac{F_0}{F_p}\right)^{2/3} = K \; (1+\gamma) \; \left(\frac{F_0}{F_d}\right)^{2/3} \tag{20}$$

with the value of K obtained from (16).

3 Governing Equations for the Underflow

According to Akiyama and Stefan (1984), since the underflow conditions determine the plunge point conditions, a model for the underflow is needed. Ellison and Turner (1959) set of governing equations for a heavy density current is used with this aim. This model is derived from basic principles in a different set of lecture notes. Here, the resulting depth-averaged governing equations are just cited. The variation of the density current height, h_d, along the bottom (direction x), is given by:

$$\frac{dh_d}{dx} = E + \frac{h_d}{3 \; Ri} \; \frac{dRi}{dx} \tag{21}$$

where E is the entrainment rate of ambient water into the density current, due interfacial mixing along the current, and Ri is the current Richardson number defined as:

$$Ri = \frac{\epsilon_d \; g \; h_d^3}{q_d^2} = \frac{1}{F_d^2} \tag{22}$$

Ellison and Turner's model also yields the following two equations:

$$\frac{dh_d}{dx} = \frac{(2 - S_1 \; Ri/2) \; E - S_2 \; Ri \; S + f_t}{1 - S_1 \; Ri} \tag{23}$$

$$\frac{h_d}{3 \; Ri} \; \frac{dRi}{dx} = \frac{(1 + S_1 \; Ri/2) \; E - S_2 \; Ri \; S + f_t}{1 - S_1 \; Ri} \tag{24}$$

where S is the bottom slope, f_t is the total (interfacial plus bottom) friction factor, and S_1 and S_2 are shape factors resulting from the depth-averaging procedure (more details are presented in a separate set of lecture notes). According to Ellison and Turner, these coefficients have the values: $S_1 \approx 0.2 - 0.3$; $S_2 \approx 0.6 - 0.9$.

The entrainment rate, E, is a function of the Richardson Number. Ashida and Egashira (1977) propose:

$$E = \frac{\beta}{Ri} \tag{25}$$

with $\beta \approx 0.0015$.

It is possible to identify two different characteristic underflow conditions from the density current model equations, (21) to (24). The first one corresponds to *critical flow*, when $dh_d/dx \to \pm\infty$, occurring, from (23), for:

$$Ri = Ri_c = \frac{1}{S_1} \tag{26}$$

where Ri_c denotes *critical Richardson number*. The second one corresponds to *normal flow*, when $dRi/dx = 0$. In this case $Ri = Ri_n$, where Ri_n denotes *normal Richardson number*. Since for this flow Ri_n is constant, so is the entrainment rate, E, therefore (23) reduces to:

$$\frac{dh_d}{dx} = E = \frac{\beta}{Ri_n} \tag{27}$$

which implies that the underflow height increases linearly along the bottom due to ambient water entrainment caused by interfacial mixing.

Imposing $dRi/dx = 0$ in (24) yields:

$$(1 + S_1\ Ri/2)\ E - S_2\ Ri\ S + f_t = 0 \tag{28}$$

from where a quadratic equation for Ri_n is obtained:

$$Ri_n^2 - \frac{1}{S_2\ S}\ (f_t + \frac{S_1\ \beta}{2})\ Ri_n - \frac{\beta}{S_2\ S} = 0 \tag{29}$$

with the solution:

$$Ri_n = \frac{1}{2\ S_2\ S}\ \{f_t + \frac{S_1\ \beta}{2} + \sqrt{\left(f_t + \frac{S_1\ \beta}{2}\right)^2 + 4\ S_2\ S\ \beta}\ \} \tag{30}$$

The bottom slope controls the type of underflow. The normal Richardson number is related to the bottom slope S through:

$$S = \frac{f_t\ Ri_n + (1 + S_1\ Ri_n/2)\ \beta}{S_2 Ri_n^2} \tag{31}$$

The *critical slope*, S_c is obtained when the normal Richardson number is equal to the critical Richardson number, $Ri_n = Ri_c = 1/S_1$:

$$S_c = \frac{S_1\ f_t + 3/2\ \beta\ S_1^2}{S_2} \tag{32}$$

According to Akiyama and Stefan, with order of magnitude considerations this equation can be reduced to:

$$S_c = \frac{S_1\ f_t}{S_2} \tag{33}$$

and using the following approximated values: $S_1 \approx 0.25$, $S_2 \approx 0.75$ and $f_t = 0.02$, yields: $S_c \approx 0.007$.

From this analysis, two different conditions can be identified: a *mild slope*, for which $S < S_c$, and a *steep slope*, for which $S > S_c$. Thus, depending on the value of S the underflow can be *supercritical* or *subcritical*. The subcritical flow (mild slope) is controlled from downstream and the height h_d that determines the plunge point conditions corresponds to normal flow conditions ($F_d^2 = 1/Ri_n$). On the contrary, the supercritical flow (steep slope) is controlled from upstream, and the underflow height immediately downstream from the plunge point, h_d, should be determined by the critical flow conditions: ($F_d^2 = 1/Ri_c$).

4 Prediction of Plunge Point Conditions

The previous analysis gives values of F_d^2 needed to predict plunge point conditions according to (19) and (20). The solution depends on the value of the bottom slope:

Mild Slope, $S < S_c$:

In this case, the normal Richardson number given by (30) can be approximated by:

$$Ri_n \approx \frac{f_t}{S_2\ S} \tag{34}$$

since for subcritical flow Ri values should be rather large. The underflow densimetric Froude number is then given by:

$$F_d^2 = \frac{1}{Ri_n} = \frac{S_2\ S}{f_t} \tag{35}$$

and therefore, from (19) and (20):

$$F_p^2 = \frac{S_2\ S}{f_t\ (1+\gamma)^3\ K^3} \tag{36}$$

$$\frac{h_p}{h_0} = K\ (1+\gamma)\ F_0^{2/3} \left(\frac{f_t}{S_2\ S}\right)^{1/3} \tag{37}$$

with the value of K given by (16) using F_d given by (35).

Steep Slope, $S > S_c$:

In this case:

$$F_d^2 = \frac{1}{Ri_c} = S_1 \tag{38}$$

and therefore, from (19) and (20):

$$F_p^2 = \frac{S_1}{(1+\gamma)^3\ K^3} \tag{39}$$

$$\frac{h_p}{h_0} = K\ (1+\gamma)\ F_0^{2/3}\ \frac{1}{S_1^{1/3}} \tag{40}$$

with the value of K given by (16) using F_d given by (38).

In order to predict the plunge point conditions the value of the mixing coefficient γ needs to be specified. However, values of γ are highly variable. Values obtained from experimental studies tend to be larger than those observed in the field. According to Akiyama and Stefan (1981), γ values would range from 0 to 1 in natural reservoirs with a mild slope. In an experimental study reported by Akiyama and Stefan (1987) values of γ were in the range from 0 to about 0.3 for mild slope conditions.

Assuming negligible plunging mixing in mild slope conditions leads to $\gamma = 0$, $K = 1$, and:

$$F_p^2 = \frac{S_2 \; S}{f_t} \tag{41}$$

$$\frac{h_p}{h_0} = F_0^{2/3} \; \left(\frac{f_t}{S_2 \; S} \right)^{1/3} \tag{42}$$

For larger values of γ, more information is required regarding the values of the coefficients S_1, S_2 and f_t. In fact, the restriction (18), imposed in order to have values of $K \geq 1$, implies that values of F_d^2 cannot be smaller than 0.25 if $\gamma > 0$. This is incompatible with solutions (35) and (38) for the values of $S_1 \approx 0.2 - 0.3$, $S_2 \approx 0.6 - 0.9$, and $f_t \approx 0.02$, suggested by Akiyama and Stefan (1984), for a wide range of slopes and not so large values of γ (e.g., lower than about 0.15). Apparently, this aspect of Akiyama and Stefan's model needs more research.

Other available predictive relationships for plunging conditions are:

Elder and Wunderlich (1973)

$$F_p = 0.5 \tag{43}$$

relationship calibrated using field data from Fontana reservoir.

Savage and Bringberg (1975):

$$F_p = \frac{2.05}{1 + f_i/f_b} \; \left(\frac{S}{f_b} \right)^{0.478} \tag{44}$$

where f_i and f_b denote the interfacial and bottom friction coefficients, respectively.

Jain (1978):

$$F_p = 0.494 \; \left(\frac{1 + f_i/f_b}{f_i/f_b} \right)^{0.189} \; \left(\frac{f_t}{8 \; S} \right)^{0.012} \tag{45}$$

Hebbert et al. (1979):

$$h_p = 1.16 \; \left(\frac{Q^2}{\epsilon_0 \; g} \right)^{1/5} \tag{46}$$

where Q is the total inflow discharge. This relationship was derived assuming a triangular instead of a rectangular cross section and was calibrated with the aid of field data from Wellington reservoir.

Finally, Akiyama and Stefan (1987) found that in a diverging horizontal channel no unique F_p value exists. Their experimental data yielded F_p values in the range from about 0.56 to 0.89, with a mean value of 0.68.

5 References

- Akiyama, J. and Stefan, H. (1981) Theory of plunging flow into a reservoir. Internal Memorandum No. IM-97. Saint Anthony Falls Hydraulic Laboratory. University of Minnesota.
- Akiyama, J. and Stefan, H. (1984) Plunging flow into a reservoir: Theory. Journal of Hydraulic Engineering. Vol. 110, No. 4, pp. 484-499.
- Akiyama, J. and Stefan, H. (1987) Onset of underflow in slightly diverging channels. Journal of Hydraulic Engineering. Vol. 113, No. 7, pp. 825-844.
- Ashida, K. and Egashira, S. (1977) Basic study on turbidity currents. Proceedings Japan Society of Civil Engineers. No. 237, pp. 37-50.
- Elder, R.A. and Wunderlich, W.O. (1973) Inflow density currents in TVA reservoirs. Proceedings of the International Symposium on Stratified Flow, IAHR, Novosibirsk.
- Ellison, T.H. and Turner, J.S. (1959) Turbulent entrainment in stratified flows. Journal of Fluid Mechanics. Vol. 6, pp. 423-448.
- Hebbert, B., Imberger, J., Loh, I. and Patterson, J. (1979) Collie river underflow into the Wellington reservoir. Journal of the Hydraulics Division, ASCE. Vol. 105, HY5, pp. 533-545.
- Jain, S.C. (1981) Plunging phenomenon in reservoirs. Proceedings of the Symposium on Surface Water Impoundments. Minneapolis.
- Savage, S.B. and Brimberg, J. (1975) Analysis of plunging phenomena in water resources. Journal of Hydraulic Research. Vol. 13, No. 2, pp. 187-204.