CORIOLIS EFFECT ON SURFACE AND INTERNAL WAVES
Linear Theory for Long Waves of Small Amplitude

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Free oscillations of one- or two-layer water basins have been analyzed in previous notes. In the analysis, the effect of the Coriolis force induced by the rotation of the earth was neglected. It is shown here that such simplification is valid as long as the wavelengths considered are small enough. Considering that the longest possible wavelength in a given water basin is of a scale similar to its horizontal dimensions, then, in a given lake or reservoir, Coriolis effects can be neglected only if those dimensions are small enough. It is also shown in what follows, that the dimensions for which Coriolis effects cannot be neglected decrease substantially in the case of internal oscillations in a stratified water body with respect to the non-stratified case. Finally, it is demonstrated here that Coriolis effects give rise to two different types of waves, known as *Poincaré* and *Kelvin waves*, respectively.

The problem of free oscillations influenced by Coriolis force in a non-stratified, one-layer water basin is analyzed first.

Case 1: Non-stratified water basin

Consider a water basin with a depth h_1 and a constant and uniform density ρ_1. The linear shallow water wave equations derived in previous notes are used, consisting of depth-averaged Reynolds-averaged Navier-Stokes equations, which are further simplified by linearizing the non-linear advection terms. Since only free oscillations are considered, it is assumed that no forcing coming from surface or bottom shear stresses exists. On the other hand, the governing equations must include the effect of the Coriolis force induced by the rotation of the earth. This imposes the need for a two-dimensional analysis, in the horizontal $x-y$ plane, since Coriolis force induces horizontal circulations in the water basin. Therefore, the governing equations for the one-layer water basin are three: momentum equations in the horizontal plane, and continuity equation for the horizontal velocity components, all depth-averaged, linearized and accounting for Coriolis force.

With this considerations, the system of equations governing the Coriolis influenced oscillations in the water basin is given by:

$$\frac{\partial U_1}{\partial t} + g\frac{\partial \xi_1}{\partial x} - fV_1 = 0 \tag{1}$$

$$\frac{\partial V_1}{\partial t} + g\frac{\partial \xi_1}{\partial y} + fU_1 = 0 \tag{2}$$

$$\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y} + \frac{1}{h_1}\frac{\partial \xi_1}{\partial t} = 0 \tag{3}$$

where U_1 and V_1 denote the horizontal components in directions x and y, respectively, of the depth-averaged velocities of the flow associated with the oscillations of the system, and $\xi_1(x, y)$ is the vertical displacement of the free surface associated with those oscillations.

In the above system of equations f represents the *Coriolis parameter*, *Coriolis frequency* or simply *inertial frequency*, which is positive in the northern hemisphere and negative in the southern hemisphere. It can be shown that this parameter is given by:

$$f = 2\Omega \sin\theta \tag{4}$$

where Ω denotes the angular velocity of the earth and θ the latitude of the point upon which f is determined. Since $\Omega = 2\pi/day = 7.27 \times 10^{-5}\ s^{-1}$, then, according to (4), f varies from a value of $\pm 1.45 \times 10^{-4}\ s^{-1}$ at the poles to zero at the equator.

In order to determine the normal modes of oscillation, assume a progressive wave type of response, for which the mean flow velocity and the free surface deformation are given by:

$$U_1 = \Upsilon_1 \exp(i(\alpha x + \beta y - \omega t)) \tag{5}$$

$$V_1 = \Psi_1 \exp(i(\alpha x + \beta y - \omega t)) \tag{6}$$

$$\xi_1 = \Xi_1 \exp(i(\alpha x + \beta y - \omega t)) \tag{7}$$

where $\alpha = 2\pi/\lambda_x$, $\beta = 2\pi/\lambda_y$ and $\omega = 2\pi/T$, and λ_x, λ_y and T denote the wavelengths in the horizontal directions x and y, and the period of the waves, respectively. Likewise, Υ_1, Ψ_1 and Ξ_1 denote the amplitudes of the velocity waves in the directions x and y and of the free surface deformation wave, respectively.

In equations (5) to (7) the identity $\exp(ia) = \cos a + i\ \sin a$ has been used, for convenience, to work with the progressive waves. Rigorously, expressions (5) to (7) involve complex numbers. To avoid that, the respective complex conjugate should be added to each expression. In practice, however, the algebra resulting from neglecting the complex conjugate is much simpler than when such complex is considered in the analysis and the final result is exactly the same, when only the real part of the solution is considered.

Replacing equations (5) to (7) in the system (1)-(3), the following algebraic problem for Υ_1, Ψ_1 and Ξ_1 is obtained.

$$\begin{vmatrix} -i\omega & -f & i\alpha g \\ f & -i\omega & i\beta g \\ i\alpha & i\beta & -i\frac{\omega}{h_1} \end{vmatrix} \begin{vmatrix} \Upsilon_1 \\ \Psi_1 \\ \Xi_1 \end{vmatrix} = 0 \tag{8}$$

This is an eigenvalue problem. In order for a non-trivial solution for the amplitudes Υ_1, Ψ_1 and Ξ_1 to exist, the matrix of coefficients must have a zero determinant. This condition yields the dispersion relationship for the oscillations of the system, i.e., the relationship between the period T and given wavelengths λ_x and λ_y, as:

$$-i\alpha^2 g\omega - i\beta^2 g\omega - i\frac{f^2\omega}{h_1} + i\frac{\omega^3}{h_1} = 0 \tag{9}$$

Calling $K^2 = \alpha^2 + \beta^2$, where K is the composite wavenumber, then dispersion relationship in this case is:

$$K^2 g h_1 + f^2 = \omega^2 \tag{10}$$

which gives:

$$\frac{\omega}{K} = \pm \frac{\sqrt{g h_1}}{\sqrt{1-(f/\omega)^2}} \tag{11}$$

If $f/\omega << 1$ then the result $c = \omega/K = \pm\sqrt{g h_1}$ for the celerity of the surface waves neglecting Coriolis effects is recovered. In this case K represents simply the wavenumber in the direction of propagation of the waves. Calling $\lambda = 2\pi/K$ the respective wavelength, then the condition $f/\omega << 1$ can be written as:

$$\frac{\lambda}{2\pi} << \frac{c}{f} \tag{12}$$

where $c = \sqrt{g h_1}$. This relationship indicates that in order to neglect Coriolis effects, the wavelength considered must be smaller than approximately c/f. This ratio is known as *Rossby radius of deformation*. Assuming that the largest wavelength that can take place in a given lake or reservoir is similar to its horizontal dimensions, then from this result it is concluded that in order to neglect Coriolis effects in a given water body, its horizontal extension must be smaller than Rossby radius of deformation. The *Burger number* (Antenucci and Imberger, 2001) is defined as:

$$Bu = \frac{c/f}{L} \tag{13}$$

where L is a length scale characterizing the horizontal extension of the water body. Hence, Coriolis effects are important only if the Burger number is smaller than about unity.

On the other hand, if f/ω is not negligible, then the celerity of the surface waves is clearly modified due to Coriolis effect. Waves affected by Coriolis are called *Poincaré waves*, *Sverdrup waves*, o simply, *rotational gravitational waves*. Equation (10) indicates that these waves can propagate in any horizontal direction and follow the relation $\omega > f$, that is, they are *superinertial* waves (Csanady, 1967).

The symmetry of the dispersion relation (10) with respect to α and β means that the directions x and y do not influence the wave field. This means that, actually, the x axis can be conveniently oriented in the direction of propagation of the wave, so as to have $\lambda_y = 0$ and hence $\partial\xi_1/\partial y = 0$. This implies that, for this orientation of the x axis, the wave fronts are parallel to y and therefore there is no variation of the free surface elevation along y. From equation (2) it is concluded, therefore, that the temporal variation of the transverse velocity V_1 is governed by:

$$\frac{\partial V_1}{\partial t} = -f U_1 \tag{14}$$

This equation predicts the existence of non-vanishing transverse velocities, even though the wave is propagating in direction x. This implies the existence of elliptic orbits for the fluid parcels trajectories. Indeed, replacing (5) and (6) in (14) yields:

$$\Psi_1 = -i\frac{f}{w}\Upsilon_1 \tag{15}$$

which, taking only the real part of the equations, yields:

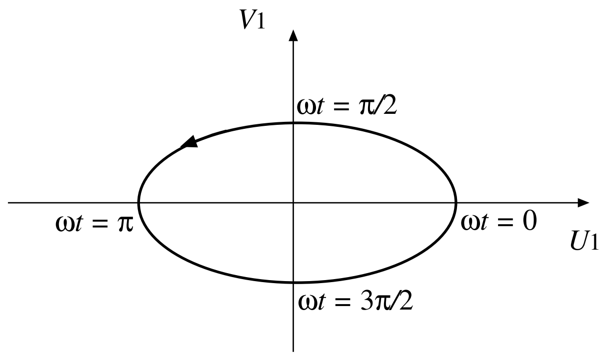

Figure 1: Orbit of a fluid parcel in a Poincaré wave in the southern hemisphere.

$$U_1 = \Upsilon_1 \cos(\alpha x - \omega t) \tag{16}$$

$$V_1 = \frac{f}{w}\Upsilon_1 \sin(\alpha x - \omega t) \tag{17}$$

From these equations it is concluded that the major axis of the orbits is oriented in the direction of propagation of the waves, and the rotation is clockwise in the northern hemisphere, since $f > 0$ in that case, and counterclockwise in the south, as $f < 0$ there (Fig. 1). As the ratio f/ω decreases, the elliptical orbits get narrower and narrower, approximating the rectilinear trajectory of the fluid parcels that is characteristic of gravitational waves not affected by Coriolis, for which $V_1 = 0$.

Consider now a situation for which the waves are affected by the boundaries of the water body. In particular, consider a wave propagating in direction x, parallel to a solid wall. The presence of the wall imposes a non-zero pressure gradient $\partial\xi_1/\partial y$, which nonetheless can relax to zero at a certain distance normal to the wall. If in this case the transverse pressure gradient is balanced by Coriolis force, fU_1, then a gravitational wave with zero transverse velocity V_1 and, consequently, with rectilinear fluid parcels trajectories, is obtained.

Imposing the condition $V_1 = 0$ in the system of equations (1)-(3) yields:

$$\frac{\partial U_1}{\partial t} + g\frac{\partial \xi_1}{\partial x} = 0 \tag{18}$$

$$g\frac{\partial \xi_1}{\partial y} + fU_1 = 0 \tag{19}$$

$$\frac{\partial U_1}{\partial x} + \frac{1}{h_1}\frac{\partial \xi_1}{\partial t} = 0 \tag{20}$$

Assume a solution given by a progressive wave in the direction x but with an unknown structure in direction y:

$$U_1 = \Upsilon_1(y)\exp(i(\alpha x - \omega t)) \tag{21}$$

$$\xi_1 = \Xi_1(y)\exp(i(\alpha x - \omega t)) \tag{22}$$

Replacing these in the system of equations (18)-(20) yields:

$$-i\omega\Upsilon_1 + ig\alpha\Xi_1 = 0 \tag{23}$$

$$g\frac{\partial \Xi_1}{\partial y} + f\Upsilon_1 = 0 \tag{24}$$

$$i\alpha\Upsilon_1 - i\frac{\omega}{h_1}\Xi_1 = 0 \tag{25}$$

From equations (23) and (25) Ξ_1 is given by:

$$\Xi_1(\alpha^2 g h_1 - \omega^2) = 0 \tag{26}$$

so in order for a non-trivial solution for Ξ_1 to exist, it is required that:

$$\frac{\omega^2}{\alpha^2} = gh_1 \tag{27}$$

which gives:

$$c = \frac{\omega}{\alpha} = \pm\sqrt{gh_1} \tag{28}$$

which is the celerity of the progressive wave considered in this new case, and which is identical to the celerity of gravity waves not affected by Coriolis.

To determine the transverse structure of the wave, equation (24) is used in combination with (25), which yields:

$$\frac{\partial \Xi_1}{\partial y} \pm \frac{f}{c}\,\Xi_1 = 0 \tag{29}$$

with c given by (27). Obviously, the only solution of this equation that makes physical sense is that for which the amplitude of the wave decays away from the coast, i.e.:

$$\Xi_1 = \Xi_{10}\ \exp(-\frac{f}{c}y) \tag{30}$$

where Ξ_{10} denotes the amplitude of the wave at the coast ($y = 0$).

Replacing this result in (25) and using (27) yields:

$$\Upsilon_1 = \Xi_{10}\sqrt{\frac{g}{h_1}}\ \exp(-\frac{f}{c}y) \tag{31}$$

So, replacing back in (21) and (22), the following relationships for the mean velocity and the surface deformation are finally obtained:

$$U_1 = \Xi_{10}\sqrt{\frac{g}{h_1}}\ \exp(-\frac{f}{c}y)\ \cos(\alpha x - \omega t) \tag{32}$$

$$\xi_1 = \Xi_{10}\ \exp(-\frac{f}{c}y)\ \cos(\alpha x - \omega t) \tag{33}$$

where only the real part of equations (21) and (22) has been considered.

Equations (32) and (33) show an exponential decay of the wave amplitude in direction y , at a rate given by the Rossby radius of deformation: f/c (Fig. 2).

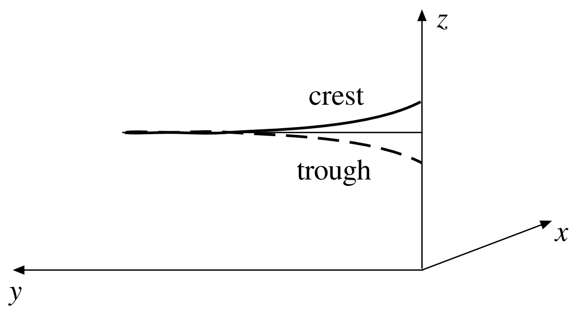

Figure 2: Transverse profile of a Kelvin wave propagating in direction x.

The waves resulting from this analysis are called *Kelvin waves*. Therefore, Poincaré and Kelvin waves are two possible and different responses of the water body to the Coriolis effect. Fig. 3 shows the dispersion relationship obtained from the present analysis for both wave types. Note that while Poincaré waves are always superinertial ($\omega > f$), Kelvin waves can be both subinertial ($\omega < f$) or superinertial.

Case 2: Two-layer stratified water basin

In this case it is necessary to apply the linearized depth-averaged RANS equations with Coriolis effects to each layer, in both horizontal directions x and y. A system of six equations is obtained:

$$\frac{\partial U_1}{\partial t} + g\frac{\partial \xi_1}{\partial x} - fV_1 = 0 \tag{34}$$

$$\frac{\partial V_1}{\partial t} + g\frac{\partial \xi_1}{\partial y} + fU_1 = 0 \tag{35}$$

$$\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y} + \frac{1}{h_1}\frac{\partial \xi_1}{\partial t} - \frac{1}{h_1}\frac{\partial \xi_2}{\partial t} = 0 \tag{36}$$

$$\frac{\partial U_2}{\partial t} + g\frac{\rho_1}{\rho_2}\frac{\partial \xi_1}{\partial x} + g\frac{(\rho_2 - \rho_1)}{\rho_2}\frac{\partial \xi_2}{\partial x} - fV_2 = 0 \tag{37}$$

$$\frac{\partial V_2}{\partial t} + g\frac{\rho_1}{\rho_2}\frac{\partial \xi_1}{\partial y} + g\frac{(\rho_2 - \rho_1)}{\rho_2}\frac{\partial \xi_2}{\partial y} + fU_2 = 0 \tag{38}$$

$$\frac{\partial U_2}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{1}{h_2}\frac{\partial \xi_2}{\partial t} = 0 \tag{39}$$

where U_i and V_i denote the depth-averaged velocity components in directions x and y, respectively, in each layer i, with $i = 1$ denoting the surface layer and $i = 2$ the bottom layer, of the flow associated with the oscillations of the system, and $\xi_1(x, y)$ and $\xi_2(x, y)$ denote the vertical displacement of the free surface and density interface, respectively, associated with those oscillations. Likewise, h_1 and h_2 denote the thicknesses of the surface and bottom layers, respectively, and ρ_1 and ρ_2 denote the water density in those same layers, respectively.

To determine the normal modes of oscillation of the Poincaré waves in the two-layer system, a progressive wave type of response is assumed, where the depth-averaged velocity components in each layer and the deformation of the free surface and density interface are given by:

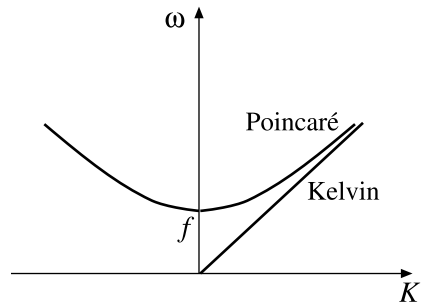

Figure 3: Dispersion relationship for Poincaré and Kelvin waves.

$$U_i = \Upsilon_i \exp(i(\alpha x + \beta y - \omega t)) \tag{40}$$

$$V_i = \Psi_i \exp(i(\alpha x + \beta y - \omega t)) \tag{41}$$

$$\xi_i = \Xi_i \exp(i(\alpha x + \beta y - \omega t)) \tag{42}$$

where $i = 1, 2$ denote surface or bottom layers, respectively. Just as in the non-stratified case, $\alpha = 2\pi/\lambda_x$, $\beta = 2\pi/\lambda_y$, $\omega = 2\pi/T$, and λ_x, λ_y and T denote the wavelengths and period of the oscillations, respectively. Likewise, Υ_i, Ψ_i and Ξ_i denote the amplitudes of the velocity waves in each layer i and surface and interface deformation waves, respectively. Replacing these expressions in the system of equations (34)-(39), the following algebraic problem for Υ_i, Ψ_i and Ξ_i is obtained:

$$\begin{vmatrix} -i\omega & -f & i\alpha g & 0 & 0 & 0 \\ f & -i\omega & i\beta g & 0 & 0 & 0 \\ i\alpha & i\beta & -i\frac{\omega}{h_1} & 0 & 0 & i\frac{\omega}{h_1} \\ 0 & 0 & i\alpha g\frac{\rho_1}{\rho_2} & -i\omega & -f & i\alpha g\frac{(\rho_2-\rho_1)}{\rho_2} \\ 0 & 0 & i\beta g\frac{\rho_1}{\rho_2} & f & -i\omega & i\beta g\frac{(\rho_2-\rho_1)}{\rho_2} \\ 0 & 0 & 0 & i\alpha & i\beta & -i\frac{\omega}{h_2} \end{vmatrix} \begin{vmatrix} \Upsilon_1 \\ \Psi_1 \\ \Xi_1 \\ \Upsilon_2 \\ \Psi_2 \\ \Xi_2 \end{vmatrix} = 0 \tag{43}$$

This is, again, an eigenvalue problem. In order for a non-trivial solution for the amplitudes Υ_i, Ψ_i and Ξ_i to exist, the coefficient matrix must have a determinant equal to zero. Imposing such condition yields:

$$-\frac{f^4\omega^2}{h_1h_2} + 2\frac{f^2\omega^4}{h_1h_2} - \frac{\omega^6}{h_1h_2} + (\alpha^2+\beta^2)^2 g^2\omega^2(\frac{\rho_1}{\rho_2} - 1) +$$
$$(\alpha^2+\beta^2)(-f^2 g\omega^2 + g\omega^4)(\frac{1}{h_1} + \frac{1}{h_2}) = 0 \tag{44}$$

Simplifying the above equation, introducing the composite wave number: $K^2 = \alpha^2 + \beta^2$ and the definition: $c_i^2 = \frac{(\rho_2-\rho_1)}{\rho_2}\ g\ \frac{h_1h_2}{h_1+h_2}$, gives:

$$\frac{K^2}{c_i^2}\ (\omega^2 - f^2) - K^4 - \frac{(\omega^2-f^2)^2}{c_i^2 g(h_1+h_2)} = 0 \tag{45}$$

equation that represents the dispersion relationship for long waves of small amplitude in a two-layer stratified water basin affected by Coriolis force (*internal Poincaré waves*). It is convenient to note that the third term of the left hand side of the equation is of a lesser order of magnitude than the rest of the terms in the equation. Taking this into consideration, it is easy to verify that, without important errors, equation (45) can be approximated by the following relationship:

$$K^2 c_i^2 + f^2 = \omega^2 \tag{46}$$

which gives:

$$\frac{\omega}{K} = \pm \frac{c_i}{\sqrt{1 - (f/\omega)^2}} \tag{47}$$

If $f/\omega << 1$ then the celerity of internal waves not affected by Coriolis is recovered: $\omega/K = \pm c_i$. In this case K simply represents the wave number in the direction of propagation of the waves. Calling $\lambda = 2\pi/K$ the corresponding wavelength, then the condition $f/\omega << 1$ can be written as:

$$\frac{\lambda}{2\pi} << \frac{c_i}{f} \tag{48}$$

Since c_i, according to its definition, corresponds to the celerity of internal gravity waves not affected by Coriolis in a two-layer stratified water basin, then it is concluded that in order to neglect Coriolis effects upon internal Poincaré waves, the wavelength considered must be lower than about c_i/f. This ratio is known as *internal Rossby radius of deformation*. Assuming that the longest wavelength that can occur in a given lake or reservoir is similar to its horizontal dimensions, then to neglect Coriolis effects on the internal waves those dimensions must be smaller than the internal Rossby radius of deformation. This condition can be expressed in terms of an internal Burger number (Antenucci and Imberger, 2001):

$$Bu_i = \frac{c_i/f}{L} \tag{49}$$

where, again, L is a length scale characterizing the horizontal extension of the water body. Hence, for Coriolis effects to be important in the two-layer stratified water basin, the internal Burger number must be smaller than about unity. Since c_i is clearly smaller than c, the internal Burger number, Bu_i, is obviously smaller than Bu, and the dimensions required for the stratified water body not to be affected by Coriolis are reduced considerably with respect to the non-stratified situation.

The symmetry of the dispersion relation (47) with respect to α and β implies that the x axis can be conveniently oriented in the direction of propagation of the wave, so that $\lambda_y = 0$, and $\partial\xi_i/\partial y = 0$. This means that for this orientation of the x axis, wave fronts are parallel to y and therefore there is no variation of the elevation of the free surface and density interface along y. From (35) and (38) it is thus concluded that, in this case, the time variation of the transverse velocities V_1 and V_2 is governed by the relationship:

$$\frac{\partial V_i}{\partial t} = -f U_i \tag{50}$$

with $i = 1, 2$. Just as in the previously studied non-stratified case, this equation predicts the existence of non-zero transverse velocities in the system, even though the waves propagate unidirectionally along x, which indicates the existence of elliptic orbits for the trajectories of fluid parcels in the water basin, just as it was shown for the non-stratified case.

Consider now Kelvin waves, which are affected by the boundaries of the water basin. For waves propagating in the direction x, parallel to a solid wall, the wall imposes a non-zero transverse pressure gradient $\partial\xi_i/\partial y$, which nevertheless vanishes at some distance away from it. If in this case the transverse pressure gradient in each layer is balanced by the Coriolis force in each layer, fU_i, then the surface and internal waves in the system will have a zero transverse velocity, V_i, and consequently fluid parcels will have rectilinear trajectories.

Imposing the condition $V_i = 0$ in the system of equations (34)-(39), yields:

$$\frac{\partial U_1}{\partial t} + g\frac{\partial \xi_1}{\partial x} = 0 \tag{51}$$

$$g\frac{\partial \xi_1}{\partial y} + fU_1 = 0 \tag{52}$$

$$\frac{\partial U_1}{\partial x} + \frac{1}{h_1}\frac{\partial \xi_1}{\partial t} - \frac{1}{h_1}\frac{\partial \xi_2}{\partial t} = 0 \tag{53}$$

$$\frac{\partial U_2}{\partial t} + g\frac{\rho_1}{\rho_2}\frac{\partial \xi_1}{\partial x} + g\frac{(\rho_2-\rho_1)}{\rho_2}\frac{\partial \xi_2}{\partial x} = 0 \tag{54}$$

$$g\frac{\rho_1}{\rho_2}\frac{\partial \xi_1}{\partial y} + g\frac{(\rho_2-\rho_1)}{\rho_2}\frac{\partial \xi_2}{\partial y} + fU_2 = 0 \tag{55}$$

$$\frac{\partial U_2}{\partial x} + \frac{1}{h_2}\frac{\partial \xi_2}{\partial t} = 0 \tag{56}$$

Consider, just as in the previous section for a non-stratified water body, a solution for the Kelvin waves consisting of a progressive wave in direction x, but with an unknown structure in direction y:

$$U_i = \Upsilon_i(y)\exp(i(\alpha x - \omega t)) \tag{57}$$

$$\xi_i = \Xi_i(y)\exp(i(\alpha x - \omega t)) \tag{58}$$

with $i = 1, 2$. Replacing these definitions in the system of equations (51)-(56), yields:

$$-i\omega\Upsilon_1 + i\alpha g\Xi_1 = 0 \tag{59}$$

$$g\frac{d\,\Xi_1}{dy} + f\Upsilon_1 = 0 \tag{60}$$

$$i\alpha\Upsilon_1 - i\frac{\omega}{h_1}\Xi_1 + i\frac{\omega}{h_1}\Xi_2 = 0 \tag{61}$$

$$-i\omega\Upsilon_2 + i\alpha g\frac{\rho_1}{\rho_2}\Xi_1 + i\alpha g\frac{(\rho_2-\rho_1)}{\rho_2}\Xi_2 = 0 \tag{62}$$

$$g\frac{\rho_1}{\rho_2}\frac{d\,\Xi_1}{dy} + g\frac{(\rho_2-\rho_1)}{\rho_2}\frac{d\,\Xi_2}{dy} + f\Upsilon_2 = 0 \tag{63}$$

$$i\alpha\Upsilon_2 - i\frac{\omega}{h_2}\Xi_2 = 0 \tag{64}$$

which corresponds to the expected eigenvalue problem for the amplitudes Υ_i and Ξ_i for the Kelvin waves in the stratified case. This problem results to be identical to that solved in previous lecture notes for gravity waves without Coriolis effects. Indeed, from this set of equations it is possible to obtain:

$$\begin{vmatrix} -\omega & \alpha g & 0 & 0 \\ \alpha & -\frac{\omega}{h_1} & 0 & \frac{\omega}{h_1} \\ 0 & \alpha g \frac{\rho_1}{\rho_2} & -\omega & \alpha g \frac{(\rho_2-\rho_1)}{\rho_2} \\ 0 & 0 & \alpha & -\frac{\omega}{h_2} \end{vmatrix} \begin{vmatrix} \Upsilon_1 \\ \Xi_1 \\ \Upsilon_2 \\ \Xi_2 \end{vmatrix} = 0 \tag{65}$$

which is identical to the equivalent problem of internal gravity waves in a two-layer stratified water basin not affected by Coriolis, and which has the already obtained solution for the celerity of internal waves:

$$\frac{\omega}{\alpha} = \pm\sqrt{\frac{(\rho_2-\rho_1)}{\rho_2}\,(\frac{h_1 h_2}{h_1+h_2})\, g} \tag{66}$$

which now represents the celerity of internal Kelvin waves. It is concluded that the latter have a celerity ω/α equal to that of internal gravity waves not affected by Coriolis: c_i.

The transverse structure of the Kelvin waves is obtained by solving equations (60) and (63). Using the rest of the equations of the system (59)-(64) it is easy to show that:

$$\Xi_2 = \Xi_{20}\ \exp(-\frac{f}{c_i}\, y) \tag{67}$$

$$\Xi_1 = -\frac{(\rho_2-\rho_1)}{\rho_1}\,(\frac{h_2}{h_1+h_2})\,\Xi_{20}\ \exp(-\frac{f}{c_i}\, y) \tag{68}$$

where Ξ_{20} denotes the amplitude of the internal wave at the coast ($y = 0$).

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