VERTICAL HEAT TRANSPORT IN LAKES AND RESERVOIRS

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1 Heat Transport Equation

One of the basic transport processes of heat in fluids corresponds to molecular conduction, which can be modeled by means of *Fourier's law*. This law states that the diffusive heat flux in a fluid is proportional to the temperature gradient.

Calling T the temperature, then Fourier's law can be written as:

$$\vec{f_h} = -k \, \nabla T \tag{1}$$

where $\vec{f_h}$ denotes the diffusive heat flux vector, expressed as heat per unit area per unit time, k denotes the thermal conductivity of the fluid, and the negative sign indicates that the flow of heat is from zones of high temperature towards zones of lower temperature.

The heat conservation equation applied onto a control volume of fluid, considering both advective heat fluxes due to the instantaneous velocity field and diffusive fluxes due to molecular transport of heat, and also the presence of a heat source per unit time per unit volume, W, can be written in vector form as:

$$\rho \, C_p \, \{\frac{\partial T}{\partial t} + (\vec{v} \cdot \nabla) T\} = -\nabla \cdot \vec{f_h} + W \tag{2}$$

where ρ denotes fluid density and C_p specific heat. Invoking Fourier's law this equation can be rewritten as:

$$\rho \, C_p \, \{\frac{\partial T}{\partial t} + (\vec{v} \cdot \nabla) T\} = k \, \nabla^2 \, T + W \tag{3}$$

The same equation expressed in tensor notation is:

$$\rho \, C_p \, \{\frac{\partial T}{\partial t} + u_j \, \frac{\partial T}{\partial x_j}\} = k \, \frac{\partial^2 \, T}{\partial x_j \partial x_j} + W \tag{4}$$

Assuming that the fluid is incompressible: $\nabla \cdot \vec{v} = 0$, multiplying this condition by T and adding the result to equation (4), the conservative form of the heat conservation equation is obtained:

$$\rho \, C_p \, \{\frac{\partial T}{\partial t} + \frac{\partial (u_j \, T)}{\partial x_j}\} = k \, \frac{\partial^2 \, T}{\partial x_j \partial x_j} + W \tag{5}$$

or, dividing by $\rho \, C_p$:

$$\frac{\partial T}{\partial t} + \frac{\partial (u_j\, T)}{\partial x_j} = \alpha\, \frac{\partial^2\, T}{\partial x_j \partial x_j} + \frac{W}{\rho\, C_p} \tag{6}$$

where $\alpha = k/(\rho\, C_p)$ is the thermal diffusivity of the fluid, with dimensions of length square over time.

This equation is valid for instantaneous conditions, in both laminar and turbulent flows. To analyze the turbulent flow case, characterized by fluctuations of velocity and temperature, Reynolds decomposition of both of these flow properties is introduced:

$$u_i = \bar{u_i} + u_i' \tag{7}$$

$$T = \bar{T} + T' \tag{8}$$

Introducing these decompositions in (6) and taking the ensemble average over the turbulence yields:

$$\frac{\partial \bar{T}}{\partial t} + \frac{\partial (\bar{u_j}\, \bar{T})}{\partial x_j} = \alpha\, \frac{\partial^2\, \bar{T}}{\partial x_j \partial x_j} - \frac{\partial (\overline{u_j'\, T'})}{\partial x_j} + \frac{W}{\rho\, C_p} \tag{9}$$

The second term of the right hand side represents turbulent heat fluxes, associated to the *turbulent diffusion* process. This process is analogous to molecular conduction of heat, but much more effective in terms of heat transport, since length scales of turbulent motion are much larger than those associated to molecular motion.

As with the Navier-Stokes and mass transport equations, the ensemble averaging process to eliminate fluctuating terms from the analysis leads to a closure problem, this time for the turbulent heat fluxes. Just as in the case of the Reynolds stresses, an external model is required to close the turbulent heat fluxes. By analogy with Fourier's law, a gradient model can be used, based on coefficients for turbulent diffusion of heat, α_{tj}. Such coefficients are analogous to α, but of larger magnitude, since turbulent diffusion is much more effective than molecular diffusion, as already discussed. The molecular diffusivity α is independent of direction, since molecular activity is isotropic. Turbulent diffusivities, α_{tj}, on the other hand, are direction dependent, as turbulent heat flows are typically anisotropic. The gradient model for turbulent heat fluxes is:

$$\overline{u_j'\, T'} = -\alpha_{tj}\, \frac{\partial \bar{T}}{\partial x_j} \tag{10}$$

where, in this particular case, the repeated subindex in the right hand side of the equation does not imply summation.

Replacing this model in (9) yields:

$$\frac{\partial \bar{T}}{\partial t} + \frac{\partial (\bar{u_j}\, \bar{T})}{\partial x_j} = \alpha\, \frac{\partial^2\, \bar{T}}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_j}\{\alpha_{tj}\, \frac{\partial \bar{T}}{\partial x_j}\} + \frac{W}{\rho\, C_p} \tag{11}$$

or, neglecting the effect of a variable density on α, assuming the they are small (in analogy with the Boussinesq approximation discussed in previous notes):

$$\frac{\partial \bar{T}}{\partial t} + \frac{\partial (\bar{u_j}\, \bar{T})}{\partial x_j} = \frac{\partial}{\partial x_j}\{(\alpha + \alpha_{tj})\, \frac{\partial \bar{T}}{\partial x_j}\} + \frac{W}{\rho\, C_p} \tag{12}$$

where α can be neglected in comparison with α_{tj}.

The so called *Prandtl coefficient*, σ_h, is defined as the ratio between the kinematic viscosity of the fluid, ν, and the thermal diffusion coefficient, α. Similarly, a turbulent Prandtl number, σ_{ht} can be introduced in (46) to represent the ratio between the kinematic eddy viscosity, ν_t, and the turbulent or eddy thermal diffusivity, α_t:

$$\frac{\partial \bar{T}}{\partial t} + \frac{\partial(\bar{u}_j\,\bar{T})}{\partial x_j} = \frac{\partial}{\partial x_j}\{(\frac{\nu}{\sigma_h} + \frac{\nu_t}{\sigma_{ht}})\,\frac{\partial \bar{T}}{\partial x_j}\} + \frac{W}{\rho\, C_p} \tag{13}$$

where ν_t can be estimated using any of the zero-, one-, or two-equation models described in previous notes. As it happens in the case of mass transport, turbulent thermal diffusivities resulting from this procedure are isotropic. To improve this aspect of the heat transport model requires the use of non-isotropic eddy viscosity models. Another very important aspect of modeling turbulent heat transport has to do with the fact that temperature controls fluid density, causing buoyancy effects associated with temperature variations within the flow field. Since the turbulent kinetic energy balance is affected by buoyancy, which can act as a sink or source of this energy, closure models for the eddy viscosity and the eddy thermal difussivity must be modified to take into account buoyancy effects associated to a variable density. This is discussed below.

2 Vertical Heat Transport in Lakes and Reservoirs

Vertical heat fluxes are the most important components of heat transport in lakes and reservoirs because they control temperature and hence density stratification of the water column.

In many lakes a reasonable assumption is that heat transport in horizontal planes can be neglected due to temperature homogeneity. This leads to the idea of developing an equation governing vertical variations of the horizontally averaged temperature $< \bar{T} >$, defined as:

$$< \bar{T} > (z) = \frac{1}{A}\int_A \bar{T}\, dA \tag{14}$$

where $A(z)$ denotes the horizontal area of the water body which varies with depth, and z is a vertical downwards coordinate with origin at the free surface (Fig. 1).

To obtain a governing equation for $< \bar{T} >$, equation (13) could be surface-averaged as in (14). An alternative route is to do a bulk balance of heat in a control volume defined by a horizontal slice of the lake, of area $A(z)$ and thickness dz (Fig. 1). Dropping the angular brackets and the overbar to simplify notation, T now represents the surface-averaged value of the Reynolds-averaged temperature. If f_h denotes the total vertical diffusive flux of heat (molecular and turbulent) averaged over A and W denotes a heat source term as in (2), then the heat conservation equation applied over the control volume considered yields:

$$\rho\, C_p\, A\, \frac{\partial T}{\partial t} = -\frac{\partial(A\, f_h)}{\partial z} + W\, A \tag{15}$$

Assuming that the vertical diffusive flux of heat is dominated by turbulence and invoking the gradient hypothesis (10), introducing a surface-averaged vertical thermal eddy diffusivity α_{tz}, yields:

$$\frac{\partial(A\, f_h)}{\partial z} = \rho\, C_p\, \frac{\partial(A\,\overline{w'T'})}{\partial z} = -\rho\, C_p\, \frac{\partial}{\partial z}(A\, \alpha_{tz}\, \frac{\partial T}{\partial z}) \tag{16}$$

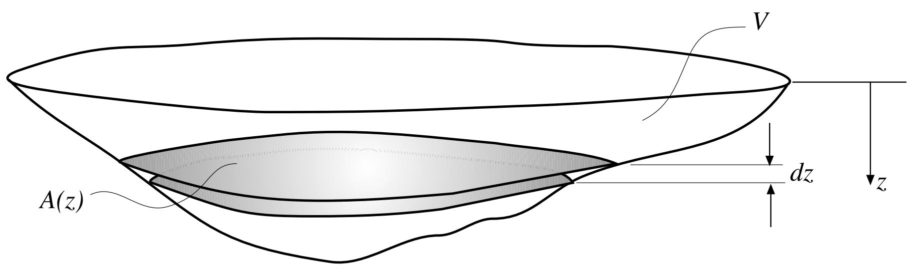

Figure 1: Element of volume of lake, $dV = A(z)\, dz$, for 1-D vertical heat balance.

where $\overline{w'T'}$ denote the surface-averaged vertical turbulent heat flux, associated to vertical velocity fluctuations w'. Replacing (16) in (15) gives:

$$\rho\, C_p\, A\, \frac{\partial T}{\partial t} = \rho\, C_p\, \frac{\partial}{\partial z}(A\, \alpha_{tz}\, \frac{\partial T}{\partial z}) + W\, A \tag{17}$$

or:

$$\frac{\partial T}{\partial t} = \frac{1}{A}\, \frac{\partial}{\partial z}(A\, \alpha_{tz}\, \frac{\partial T}{\partial z}) + \frac{W}{\rho\, C_p} \tag{18}$$

3 The Nature of the Source Term

The heat source term per unit time, per unit volume, W, is given by the short wave solar radiation that penetrates the water column. In previous notes the heat flux due to penetrative short wave radiation was estimated as:

$$H_{sw}(z) = H_{sw0}\, \exp\left(-k_e z\right) \tag{19}$$

where H_{sw0} represents the heat flux due to short wave radiation at the free surface and k_e is the light extinction coefficient.

The amount of heat transferred by short wave radiation to the control volume of Fig. 1, per unit time, is given by the net influx associated to H_{sw}, such that:

$$W\, A\, dz = -\frac{\partial (H_{sw}\, A)}{\partial z}\, dz \tag{20}$$

or:

$$W\, A = -\frac{\partial (H_{sw}\, A)}{\partial z} \tag{21}$$

Expanding this equation yields:

$$W\, A = -A\, H_{sw}\, \{\frac{1}{H_{sw}}\, \frac{\partial H_{sw}}{\partial z} + \frac{1}{A}\, \frac{\partial A}{\partial z}\} \tag{22}$$

However, it is obvious that the rate of light extinction in the water column is generally much larger than the rate of change of the transverse area of the lake. With this consideration, (22) simplifies to:

$$W = -\frac{\partial H_{sw}}{\partial z} \tag{23}$$

or, using (19):

$$W = k_e\ H_{sw} = k_e\ H_{sw0}\ \exp\left(-k_e\ z\right) \tag{24}$$

Replacing this result in (18) gives:

$$\frac{\partial T}{\partial t} = \frac{1}{A}\ \frac{\partial}{\partial z}(A\ \alpha_{tz}\ \frac{\partial T}{\partial z}) + \frac{k_e\ H_{sw0}}{\rho\ C_p}\ \exp\left(-k_e\ z\right) \tag{25}$$

4 Eddy Diffusivity

Equation (25) can be used to predict vertical temperature profiles in lakes and reservoirs, provided meteorological data is used to evaluate H_{sw} and a closure for α_{tz} is selected.

As in previous sections, a turbulent Prandtl number, σ_{ht} can be introduced to specify:

$$\alpha_{tz} = \frac{\nu_t}{\sigma_{ht}} \tag{26}$$

In many cases a value $\sigma_{ht} = 1$ is a good approximation, which is called *Reynolds analogy*. Nonetheless, when buoyancy effects caused by density variations associated to temperature gradients in the water column are important, this approximation does not hold, and a more sophisticated closure is needed.

As already discussed, buoyancy can affect the balance of turbulent kinetic energy in the water column, creating or destroying it depending on the sign of the density gradients. If the stratification is stable, vertical transport of fluid against gravity due to velocity fluctuations creates a sink of turbulent kinetic energy.

Buoyancy effects can be taken into account into the equation for the turbulent kinetic energy by considering density fluctuations. Reynolds decomposition applied to the density yields:

$$\rho = \bar{\rho} + \rho' \tag{27}$$

where $\bar{\rho}$ and ρ' denote mean and fluctuating components of the instantaneous density ρ, respectively. The density fluctuations modify the instantaneous equations for the velocity fluctuations presented in the notes of Chapter 1, (obtained by making the difference between the instantaneous and Reynolds-averaged Navier-Stokes equations), which results in additional terms that contribute to the kinetic energy balance. The resulting equation for the turbulent kinetic energy is:

$$\frac{\partial K}{\partial t} + \bar{u_j}\ \frac{\partial K}{\partial x_j} = -\frac{\partial}{\partial x_j}\{\frac{1}{\bar{\rho}}\ \overline{u'_j \hat{p}'} + \frac{1}{2}\ \overline{u'_i u'_i u'_j} - 2\nu\ \overline{u'_i \epsilon'_{ij}}\} - \overline{u'_i u'_j}\ \epsilon_{ij} + \frac{g_j}{\bar{\rho}}\ \overline{u'_j \rho'} - 2\nu\ \overline{\epsilon'_{ij}\epsilon'_{ij}} \tag{28}$$

The left hand side of this equation represent the total change of turbulent kinetic energy. The first term of the right hand side can be interpreted (and modeled) simply as turbulent diffusion of turbulent kinetic energy, the second one denotes production, P, the third one is a sink/source term, G, taking into account buoyancy effects (associated to turbulent fluxes of fluid density) and the fourth one denotes the rate of dissipation ϵ.

In the case of heat transport in the water column, density fluctuations are caused by temperature fluctuations, however, in a more general analysis, they can also be induced by fluctuations of dissolved mass concentration. Consider a linear equation of state relating fluid density and the concentration of a given dissolved species or temperature, ϕ:

$$\rho = \rho_0 \, (1 + \beta \, \phi) \tag{29}$$

where ρ denotes instantaneous density, ρ_0 is a reference density, and β is a constant coefficient, which takes different values depending on the nature of ϕ. It is easy to show that:

$$\rho' = \rho_0 \, \beta \, \phi' \tag{30}$$

and therefore the buoyancy term in (28) can be written as:

$$G = \frac{g_j}{\bar{\rho}} \, \overline{u_j' \rho'} = \frac{g_j}{\bar{\rho}} \, \rho_0 \, \beta \, \overline{u_j' \phi'} \tag{31}$$

Now assuming that the density variations, $\tilde{\rho}$, due to ϕ, with respect to the reference value ρ_0, are small (Boussinesq approximation), then:

$$\rho = \rho_0 + \tilde{\rho} = \rho_0 \, (1 + \beta \, \phi) \tag{32}$$

and:

$$\bar{\rho} = \rho_0 \, (1 + \frac{\bar{\tilde{\rho}}}{\rho_0}) \approx \rho_0 \tag{33}$$

which finally gives:

$$G = \beta \, g_j \, \overline{u_j' \phi'} \tag{34}$$

A *flux Richardson number*, R_f can be defined as minus the ratio between the rate at which turbulent kinetic energy is being spent by buoyancy and the rate of production of such energy:

$$R_f = -\frac{G}{P} = \frac{\beta \, g_j \, \overline{u_j' \phi'}}{\overline{u_i' u_j'} \, \epsilon_{ij}} \tag{35}$$

Considering a simple steady, uniform wall-bounded flow with streamwise mean velocity $\bar{u}(z)$ and vertical distribution of the species $\bar{\phi}(z)$, where z is now a vertical coordinate with origin on the wall, then production and buoyancy terms reduce to:

$$P = -\overline{u'w'} \, (\frac{\partial \bar{u}}{\partial z}) \tag{36}$$

$$G = -\beta \, g \, \overline{w' \phi'} \tag{37}$$

where w' denotes the fluctuation of the vertical component of the velocity. The flux Richardson number is therefore:

$$R_f = -\frac{G}{P} = -\beta \, g \, \frac{\overline{w' \phi'}}{\overline{u'w'} \, \frac{\partial \bar{u}}{\partial z}} \tag{38}$$

Introducing standard closures for the Reynolds stress and buoyancy fluxes:

$$\overline{u'w'} = -\nu_t \frac{\partial \bar{u}}{\partial z} \tag{39}$$

$$\overline{w'\phi'} = -D_t \frac{\partial \bar{\phi}}{\partial z} \tag{40}$$

where D_t represents the eddy diffusivity of species ϕ, yields:

$$R_f = -\beta\, g\, \frac{D_t}{\nu_t} \frac{(\partial \bar{\phi}/\partial z)}{(\partial \bar{u}/\partial z)^2} \tag{41}$$

and since from (29):

$$\frac{\partial \bar{\phi}}{\partial z} = \frac{1}{\beta\, \rho_0} \frac{\partial \bar{\rho}}{\partial z} \tag{42}$$

then the flux Richardson number reduces to:

$$R_f = -\frac{g}{\rho_0} \frac{D_t}{\nu_t} \frac{(\partial \bar{\rho}/\partial z)}{(\partial \bar{u}/\partial z)^2} \tag{43}$$

which can be simply rewritten as:

$$R_f = \frac{D_t}{\nu_t}\, Ri \tag{44}$$

since the *gradient Richardson number*, Ri, has been defined as:

$$Ri = -\frac{g}{\rho_0} \frac{(\partial \bar{\rho}/\partial z)}{(\partial \bar{u}/\partial z)^2} \tag{45}$$

Note that R_f is always positive for stable stratification.

Assuming the existence of an equilibrium region in the flow considered, such that diffusion and advection can be neglected, then production of turbulent kinetic energy must balance buoyancy and dissipation:

$$P + G - \epsilon = 0 \tag{46}$$

or, introducing the definition for R_f:

$$R_f + \frac{\epsilon}{P} = 1 \tag{47}$$

In the flow considered, the energy extracted from the mean flow goes directly to the streamwise turbulent intensity, $\overline{u'^2}$, and gets redistributed among all three components due to pressure fluctuations. The loss to buoyancy of the turbulent kinetic energy affects only $\overline{w'^2}$ directly, whereas viscosity affects all three components of the turbulent kinetic energy. From this analysis it is apparent that while buoyancy has a small effect on the energy dissipation, it may have a major impact on the turbulence structure of the stratified flow. This means also, that even in a stratified flow the rate of dissipation, ϵ, remains the dominant term in (47), from which it is concluded that the maximum possible value of R_f in a given flow is much smaller than unity. In fact, the existence of a critical value of the flux Richardson number, R_{fcrit}, has been proposed, above which the collapse of turbulence occurs. This means that vertical turbulent fluxes of the species ϕ must drop before buoyancy forces have much effect on the overall energy balance.

Ellison (1957) proposed:

$$\frac{R_f}{Ri} = \frac{D_t}{\nu_t} = b\,\frac{(1 - R_f/R_{fcrit})}{(1 - R_f)^2} \tag{48}$$

where b is a constant with a value of about unity and $R_{fcrit} \approx 0.15$. This equation predicts that the effect of buoyancy is to reduce the value of the ratio of the eddy transport coefficients, D_t/ν_t, such that this ratio tends to zero as R_f tends to $R_{fcrit} \approx 0.15$. This means that turbulent transport of the species ϕ is damped due to buoyancy effects, and that the damping is stronger as R_f increases.

Interestingly, Ellison predicts that it is the correlation $\overline{w'\rho'}$ that tends to zero as R_f tends to R_{fcrit}, and not necessarily the velocity and density fluctuations themselves. This is the case, for example, of pure internal wave motions, for which ρ' and w' are exactly 90^o out of phase, and the correlation $\overline{w'\rho'}$ vanishes.

Munk and Anderson (1948) proposed empirical formulae to estimate the effects of buoyancy on the eddy transport coefficients. These formulae use the gradient Richardson number as the parameter accounting for buoyancy and predict the damping of those coefficients as Ri increases:

$$\frac{\nu_t}{\nu_{t0}} = (1 + 10\,Ri)^{-0.5} \tag{49}$$

$$\frac{D_t}{D_{t0}} = (1 + 3.33\,Ri)^{-1.5} \tag{50}$$

where ν_{t0} and D_{t0} represent eddy viscosity and eddy diffusivity for neutral stratification ($Ri = 0$), respectively. The Prandtl/Schmidt number is thus given by:

$$\frac{\sigma_t}{\sigma_{t0}} = \frac{(1 + 3.33\,Ri)^{1.5}}{(1 + 10\,Ri)^{0.5}} \tag{51}$$

which increases as Ri increases. Here σ_{t0} denotes the value of the Prandtl/Schmidt number for neutral stratification.

These algebraic (zero-equation) relationships for the eddy transport coefficients are commonly used in connection with the mixing length model. The application of such model to estimate turbulent transport of mass or heat, however, is not recommended, because the gradient hypothesis fails even in simple cases. One of those cases corresponds, for example, to the flow in a pipe, for which the mean velocity gradient vanishes at the pipe centerline. The mixing length model predicts zero eddy viscosity and therefore no transport across this region, however, in reality, the reduction of ν_t at the centerline is much less severe than predicted by this model.

A version of the $K - \epsilon$ model corrected for buoyancy effects can be used to obtain a a better estimation of the eddy transport coefficients. The equations for K and ϵ in this case are:

$$\frac{\partial K}{\partial t} + \bar{u_j}\,\frac{\partial K}{\partial x_j} = \frac{\partial}{\partial x_j}\{\frac{\nu_t}{\sigma_K}\,\frac{\partial K}{\partial x_j}\} + P + G - \epsilon \tag{52}$$

$$\frac{\partial \epsilon}{\partial t} + \bar{u_j}\,\frac{\partial \epsilon}{\partial x_j} = \frac{\partial}{\partial x_j}\{\frac{\nu_t}{\sigma_\epsilon}\,\frac{\partial \epsilon}{\partial x_j}\} + c_{1\epsilon}\,\frac{\epsilon}{K}\,(P + G)\,(1 + c_{3\epsilon}\,R_f) - c_{2\epsilon}\,\frac{\epsilon^2}{K} \tag{53}$$

where P is the production term estimated using the eddy viscosity, ν_t, as indicated in the notes of Chapter 1, G is defined as:

$$G = \beta\, g_j \, \frac{\nu_t}{\sigma_t} \, \frac{\partial \bar{\phi}}{\partial x_j} \tag{54}$$

σ_t is the Prandtl/Schmidt number and $c_{3\epsilon}$ is an empirical constant of the model.

Rodi (1984) proposed redefining the flux Richardson number in (53) such that:

$$R_f = -\frac{G}{P+G} \tag{55}$$

to enforce the use of a unique value of $c_{3\epsilon}$ for different flow configurations.

To close this discussion it must be said that in many real applications, particularly in lakes and reservoirs, closures such as those given by the $K-\epsilon$ model do not necessarily work, in the sense that they do not totally capture the real nature of the turbulent mixing processes occurring in the water column. It is well known that in the interior of strongly stratified fluids, turbulence occurs only sporadically and in isolated patches. These patches are mainly created by the breakdown of internal waves as the result of a cascade of energy, which is transferred from the long-wave seiching of the main density interfaces within the water body, to shorter and shorter waves through non-linear interaction mechanisms. Even though the internal waves can distort the density distribution, they cannot permanently change the stratification unless they break creating turbulence and mixing. Internal waves, however, transfer energy more rapidly than turbulence, propagating the effects of boundaries to the interior of the stratified fluid. Patches of turbulence in the interior of lakes and reservoirs, therefore, can result from the superposition of motions from many sources and on many scales, the mean shear making and essential contribution to the internal breakdown mechanism.

It has been argued that models such as $K-\epsilon$ are not able to resolve the patchiness of turbulent mixing processes within water bodies such as lakes and reservoirs, mainly because of the large dimensions of the discretization grids that must be used in such domains. Besides, the model for the mean flow needs to capture internal seiching and eventually the energy cascade from large to short internal waves, something that may demand large computing efforts, making a complex turbulence closure model such as $K-\epsilon$ not viable.

A different, and somewhat heuristic, approach to the turbulent mixing problem within stratified water bodies has been proposed by Hodges et al. (2000). Their 3-D model focuses mainly in resolving internal waves and internal motions, created for example by density currents, using simple algebraic closures to account for turbulent mixing.

5 References

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