GRADIENT RICHARDSON NUMBER

CI 71Q HIDRODINAMICA AMBIENTAL   Profs. Y. Niño & A. Tamburrino
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Consider a turbulent flow with a stable stratification, as shown in Fig. 1, such that density increases downwards and the mean flow velocity (averaged over the turbulence) points in the longitudinal direction, increasing upwards. The no-slip boundary condition imposes zero velocity at the bottom wall, resulting in high velocity gradients in the near wall region and generating longitudinal shear stresses. Turbulence associated to this shear flow can induce vertical mixing in the stratified fluid, as long as the turbulent kinetic energy of the flow is large enough compared with the increase in potential energy associated to the vertical mixing of the stratified fluid. Turbulent vertical mixing causes the reduction of vertical density gradients, increasing the density of the surface waters, that is, moving the center of mass of the water column upwards and thus increasing the potential energy of the flow.

First we estimate the turbulent kinetic energy available for mixing. Call u the instantaneous longitudinal component of flow velocity at a height z_0 above the bottom wall. This velocity can be decomposed into a mean value, $\bar{u}$, and a fluctuation u', such that $u = \bar{u} + u'$. Similarly in the vertical: $w = w'$, where w is the instantaneous vertical component of the flow velocity at a height z_0 above the bottom wall and w' is the corresponding fluctuation. It is assumed that the flow has a vanishing vertical component of the mean velocity. The kinetic energy, E, associated to the vertical motion of a fluid parcel of density ρ and volume V is given by:

$$E = \frac{1}{2}\rho V (w')^2 \tag{1}$$

To estimate w' we use *Prandtl's mixing length theory*. Assume that a fluid parcel, located originally at a height z_0 above the bottom wall, undergoes a vertical displacement of length l, in

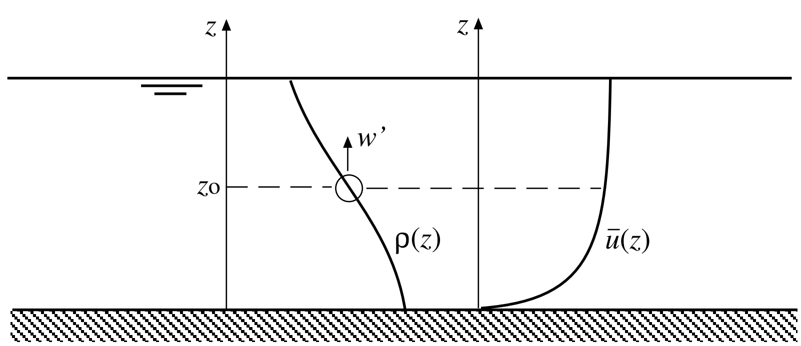

Figure 1: Turbulent flow with stable stratification.

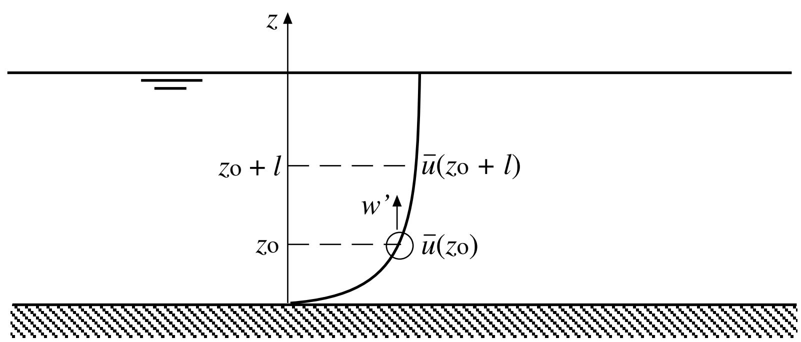

Figure 2: Turbulent fluctuation according to mixing length theory.

response to a fluctuation w' of the vertical component of the flow velocity. It can be assumed that this distance is equivalent to the *mixing length* of the flow. Based on the scheme of Fig. 2, the change in vertical position of the fluid parcel under consideration, with mean longitudinal velocity $\bar{u}(z_0)$, corresponds to a fluctuation u' of the longitudinal velocity component of the flow, since this parcel has a lower velocity relative to that of the mean flow at the new position $z_0 + l$:

$$u' = \bar{u}(z_0) - \bar{u}(z_0 + l) \tag{2}$$

Introducing a Taylor series expansion of the last term of the right hand side of the equation, yields:

$$\bar{u}(z_0 + l) = \bar{u}(z_0) + \frac{\partial \bar{u}}{\partial z}\, l + \ldots \tag{3}$$

This means that u' can be estimated as:

$$u' = -\frac{\partial \bar{u}}{\partial z}\, l \tag{4}$$

Since $\partial\bar{u}/\partial z$ is positive, this result means that u' and w' have opposite signs: positive fluctuations of the vertical velocity component generate negative fluctuations of the longitudinal velocity component and viceversa. Assuming that w' is of the same order of magnitude in absolute value than u', but of opposite sign, then:

$$w' \approx \frac{\partial \bar{u}}{\partial z}\, l \tag{5}$$

Replacing w' in (1) yields:

$$E = \frac{1}{2}\rho V (\frac{\partial \bar{u}}{\partial z}\, l)^2 \tag{6}$$

Now we estimate the increase in potential energy associated to vertical mixing. Consider the lift force, F, exerted upon the volume V, that is required to displace this volume from the initial position z_0 to a final position $z = z_0 + \Delta z$. Assuming, as a first order approximation, that the density variation is linear in the interval Δz, we have:

$$F = -\frac{d\rho}{dz}(z - z_0)gV \tag{7}$$

where the minus sign takes into account the negative sign of the stable vertical density gradient. The work required to lift the volume V from z_0 to $z_0 + l$ is then:

$$W = \int_{z_0}^{z_0+l} F dz \tag{8}$$

or using (7):

$$W = \int_{z_0}^{z_0+l} (-\frac{d\rho}{dz})(z - z_0)gV dz \tag{9}$$

Again, assuming that $d\rho/dz$ is constant in the interval z_0 to $z_0 + l$, yields:

$$W = -gV\frac{d\rho}{dz}\int_{z_0}^{z_0+l} (z - z_0)dz \tag{10}$$

or:

$$W = -gV\frac{d\rho}{dz}\frac{l^2}{2} \tag{11}$$

Taking the ratio between the work required to mix the stratified fluid in the vertical, W, given by (11), and the turbulent kinetic energy available for vertical mixing, E, given by (6), we obtain the dimensionless parameter known as *gradient Richardson number*, Ri:

$$Ri = \frac{W}{E} = -\frac{\frac{g}{\rho}\frac{d\rho}{dz}}{(\frac{\partial \bar{u}}{\partial z})^2} \tag{12}$$

This dimensionless parameter represents the ratio between the potential energy required for vertical mixing and the turbulent kinetic energy available in the flow for mixing. High values of Ri are associated to a low mixing capacity of the turbulent flow and viceversa. Usually, a limit or critical value of the gradient Richardson number, $Ri_c = 1/4$ is considered, such that in flows with larger values of Ri, the turbulence is supressed by the high density gradients and no mixing occurs.

It is interesting to examine the origin of the afore mentioned condition for Ri. Actually, it is derived from a stability analysis of inviscid continuously stratified flow. The idea is that small internal-wave like disturbances do not grow in such a flow if the gradient Richardson number is large enough. Miles (1961, 1963) demonstrated that a sufficient stability condition in this case is $Ri > 1/4$. This means that at lower values of Ri the shear flow become unstable, giving rise to turbulence and internal mixing following the breakdown of internal waves.

Considering the definition of the buoyancy frequency, N, the gradient Richardson number can also be expressed as:

$$Ri = \frac{N^2}{(\frac{\partial \bar{u}}{\partial z})^2} \tag{13}$$

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