BUOYANCY FREQUENCY
OR BRUNT-VÄISÄLÄ'S FREQUENCY

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Consider a stably stratified fluid, as shown in Fig. 1, such that its density increases downwards. Assume that ambient turbulence exists in this system, and that, at a given time, a fluid parcel located at a level z_0 above the bottom is displaced vertically and upwards due to a turbulent fluctuation of the vertical component of the fluid velocity. Due to the stable stratification, the fluid parcel will have in its new location, $z_0 + \Delta z$, a larger density than that of the ambient fluid. Due to the excess weight, the fluid parcel will tend to sink, due to inertia, to a location that is deeper than its initial location, where the ambient density is larger than that of the fluid parcel. Due to buoyancy, the particle will move upwards again, thus defining an oscillatory motion with a characteristic frequency, denoted *Brunt-Väisälä's frequency* or simply *buoyancy frequency*.

The continuity equation for a fluid with variable density is:

$$\frac{D\rho}{Dt} + \rho\,\nabla \cdot \vec{v} = 0 \tag{1}$$

where ρ denotes density, t, time and $\vec{v}$ the flow velocity vector. An incompressible fluid satisfies the condition $\nabla \cdot \vec{v} = 0$, thus the continuity equation is reduced to:

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\vec{v} \cdot \nabla)\rho = 0 \tag{2}$$

Consider now a stable density profile, $\rho_0(z)$. This constitutes a base state, which is perturbed by the presence of fluid parcels of different densities transported by the vertical component of the fluid velocity, w, associated to the oscillatory motion previously described. Calling $\rho'(z,t)$ the

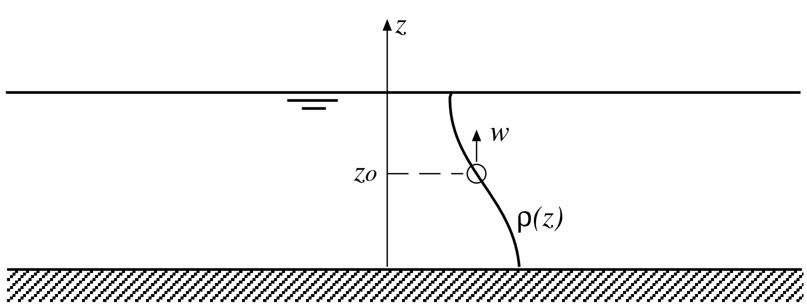

Figure 1: Fluid with stable stratification.

perturbation of the base density, then in any location z the instantaneous density, $\rho(z,t)$, is given by:

$$\rho = \rho_0 + \rho' \tag{3}$$

Since the density changes are associated only with the vertical velocity component, w, which can be considered of small magnitude, just as the density perturbation ρ' (recall that the vertical motion is driven by small density differences in the vertical), then, neglecting second order terms in (2), a linearized version of the continuity equation is obtained:

$$\frac{\partial \rho'}{\partial t} + w\frac{\partial \rho_0}{\partial z} = 0 \tag{4}$$

This equation predicts the variation in time of the density perturbation associated to the vertical oscillatory motion of the fluid.

Consider now the momentum equation governing fluid motion. Neglecting viscous effects, Euler's equation can be invoked:

$$\rho\frac{D\vec{v}}{Dt} = -\nabla p + \rho\,\vec{g} \tag{5}$$

where p, denotes thermodynamic pressure and $\vec{g}$ acceleration of gravity. Introducing Boussinesq's approximation, which considers that the density changes are important only in the mass forces (i.e., in the gravitational term of the equation) and neglects them in the inertial terms (left hand side of the equation), then the following equation is obtained:

$$\rho_0\frac{D\vec{v}}{Dt} = -\nabla p + \rho\,\vec{g} = -\nabla p - \rho\,g\,\nabla z \tag{6}$$

Assuming that the base state is in static equilibrium, such that pressure is hydrostatic, then:

$$\nabla(p_0 + \rho_0 g z) = 0 \tag{7}$$

where $p_0(z)$ denotes the pressure distribution in the base state. Defining the pressure perturbation as $p' = p - p_0$, then equation (6) can be rewritten as:

$$\rho_0\frac{D\vec{v}}{Dt} = -\nabla p' - \rho'\,g\,\nabla z \tag{8}$$

or similarly:

$$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho_0}\nabla p' - \frac{\rho'}{\rho_0}\,g\,\nabla z \tag{9}$$

This equation can be linearized, assuming small flow velocities such that:

$$\frac{\partial \vec{v}}{\partial t} = -\frac{1}{\rho_0}\nabla p' - \frac{\rho'}{\rho_0}\,g\,\nabla z \tag{10}$$

The vertical component of the equation is:

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}(\frac{\partial p'}{\partial z} + \rho' g) \tag{11}$$

Neglecting the magnitude of the pressure perturbation (which is valid as long as such perturbation is associated to small density variations) the previous equation is reduced to:

$$\frac{\partial w}{\partial t} = -\frac{\rho'}{\rho_0} g \tag{12}$$

This equation, together with equation (4), provide a solution for the velocity and density variation in time during the oscillatory motion described previously. In fact, deriving equation (12) with respect to time and replacing the result in equation (4), an equation for the evolution of the vertical velocity component, w, in time, is obtained:

$$\frac{d^2 w}{dt^2} - \frac{g}{\rho_0}\frac{d\rho_0}{dz} w = 0 \tag{13}$$

From here, the small displacements $\eta(t)$ of the fluid parcels associated to the vertical velocity component can be predicted. Considering $w = d\eta/dt$, then, integrating (4) with respect to time and replacing the result in (12) yields:

$$\frac{d^2 \eta}{dt^2} - \frac{g}{\rho_0}\frac{d\rho_0}{dz} \eta = 0 \tag{14}$$

Both equations (13) and (14) describe a simple harmonic motion, which demonstrates that the vertical displacement of fluid parcels in an otherwise stably stratified fluid indeed generates an oscillatory motion. The frequency of this motion, N, corresponds to Brunt-Väisälä's frequency or buoyancy frequency, as noted previously. According to equations (13) and (14) this frequency is given by:

$$N = \sqrt{-\frac{g}{\rho_0}\frac{d\rho_0}{dz}} \tag{15}$$

Of course, the term inside the square root is positive, since a stable stratification satisfies $d\rho_0/dz < 0$. According to this result, N is maximum within the metalimnion which is the region where the density gradients are maximum. On the contrary in the epilimnion or hypolimnion, where the density gradients are much lower, the buoyancy frequency is considerably reduced. The period of the oscillation corresponding to N is given by $2\pi/N$. Typical values of this period are of the order of a few minutes for the atmospheric or oceanic thermocline, and up to several hours in the deep ocean. This result indicates that these oscillations, or internal waves, are much more easily detected in the thermocline than in any other region of the stratified water body, from short term and high frequency temperature measurements.

An interesting case is that of an exponential variation of the density in the vertical (for instance, $\rho_0 = \rho_{0s}\, e^{-k\,(z-H)}$, where ρ_{0s} is the density in the free surface, at $z = H$, H is the water depth and k is a constant). In this case $N = g\ k$. Such kind of variation is usually considered as an approximation that simplifies the analysis of a stratified flow. Another typical approximation consists of assuming a linear variation of density in the vertical. In this case N increases downwards.

In a shear flow (or rotational flow: $\nabla \times \vec{v} \neq 0$) velocity gradients and shear stresses do exist. In such case, the vertical gradient of horizontal velocity, $\partial u/\partial z$, (associated to horizontal shear stress) has dimensions of frequency. The dimensionless ratio:

$$Ri = \frac{N^2}{(\frac{\partial u}{\partial z})^2} \tag{16}$$

is denoted *gradient Richardson number*, and represents the capacity of the shear flow vorticity to mix fluid against the density gradients, such that large values of Ri are associated to a low mixing capacity of the flow.

Reference:

Turner, J. S. (1973). Buoyancy Effects in Fluids. Cambridge University Press.