Wang and Anderson. Introduction to Groundwater Modelling.1982.

APPENDIX D

Analogies

D.1 INTRODUCTION

Physical analogies have played an important role in the development of mathematical modeling of groundwater flow. If the mathematical equations that represent two physical situations are equivalent, then the solution techniques and solutions themselves are applicable to both problems. In this appendix, three analogies to groundwater flow are described. The groundwater variables and their electrical and heat flow counterparts are summarized in Table D.1. Table D.2 is a similar summary of the structural mechanics analogy.
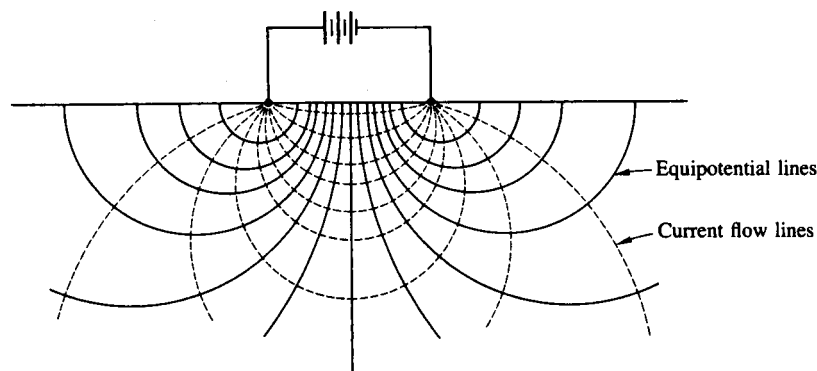
D.2 ELECTRICAL ANALOGY

M. King Hubbert's interest in the physics of groundwater flow was aroused while he was conducting electrical earth-resistivity surveys during the period from 1931 to 1936. In a resistivity survey, four electrodes are driven into the ground. A battery produces a current I through the two outer electrodes. The voltage difference V is measured across the two inner electrodes. If the sub-

Table D.1

Analogies to groundwater flow (representative units are shown in brackets)

VARIABLE	GROUNDWATER	ELECTRICITY	HEAT
Potential	Head, h [cm]	Voltage, V [Volts]	Temperature, T [°C]
Quantity transported	Volume discharge rate [cm³ s⁻¹]	Electrical charge [Coulomb]	Heat [calorie]
Physical property of medium	Hydraulic conductivity, K [cm s⁻¹]	Electrical conductivity, σ [mhos m⁻¹]	Thermal conductivity, K [cal cm⁻¹ s⁻¹ °C⁻¹]
Relation between potential and flow field	Darcy's law $\mathbf{q} = -K\,\mathbf{grad}\,h$ where q is specific discharge [cm s⁻¹]	Ohm's law $\mathbf{i} = -\sigma\,\mathbf{grad}\,V$ where i is electrical current [Amperes]	Fourier's law $\mathbf{q} = -K\,\mathbf{grad}\,T$ where q is heat flow [cal cm⁻² s⁻¹]
Storage quantity	Specific storage, S_s [cm⁻¹]	Capacitance, C [microfarad]	Heat capacity, C_v [cal cm⁻³ °C⁻¹]

surface of the earth is electrically uniform, then the theoretical lines of electrical current flow are shown by the dashed lines of Figure D.1. Perpendicular to these flow lines are the solid lines of constant voltage, that is, equipotential lines.

Equipotential lines

Current flow lines

Figure D.1

Experimental set-up for conducting an earth-resistivity survey. (After Dobrin, 1976.)

Table D.2

Structural mechanics analogy to groundwater flow

VARIABLE	GROUNDWATER	MECHANICS
Unknown variable	Head, h	Displacement, $\delta = \begin{pmatrix} u \\ v \end{pmatrix}$
First derivative quantity	$\mathbf{grad}\ h = \begin{pmatrix} \dfrac{\partial h}{\partial x} \\ \dfrac{\partial h}{\partial y} \end{pmatrix}$	$\text{Strain, } \varepsilon = \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{pmatrix}$
Conjugate variable	$\mathbf{q} = \begin{pmatrix} q_x \\ q_y \end{pmatrix}$	$\text{Stress, } \sigma = \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix}$
Physical property of medium	Hydraulic conductivity, $$[K] = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$$	Elastic moduli, $$[C] = \begin{pmatrix} \dfrac{E}{1-v^2} & \dfrac{v}{1-v^2} & 0 \\ \dfrac{v}{1-v^2} & \dfrac{E}{1-v^2} & 0 \\ 0 & 0 & G \end{pmatrix}$$ where E is Young's modulus, v is Poisson's ratio, and G is shear modulus, $G = \dfrac{E}{2(1+v)}$
Assembled finite element matrix equation	$[G]\{h\} = \{B\}$ where $[G]$ is conductance matrix and $\{B\}$ is recharge matrix	$[S]\{\delta\} = \{F\}$ where $[S]$ is stiffness matrix and $\{F\}$ is load matrix

Hubbert (1969, p. 11) asks the question: "Suppose that the ground consisted of a uniform sand covered with an impermeable layer and filled with water. Suppose further that the electrodes were to be replaced by wells terminated at the top of the sand by hemispherical screens, and the electrical battery replaced by a pump connecting the two wells. Then, with the pump operating at a

uniform rate, a water flow field would be created through the sand between the two wells. What would be the nature of the flow field, and what would be the equations describing the flow?"

Hubbert intuitively felt that the water flow lines would be the same as the electrical current flow lines. The analogy is exact in the sense that mathematically identical equations describe both cases. For a particular material, the current I is directly proportional to voltage drop $V_2 - V_1$ and cross-sectional area A, but inversely proportional to length $\ell_2 - \ell_1$. Calling the proportionality constant the electrical conductivity σ gives the relationship

$$I = -\sigma A \frac{V_2 - V_1}{\ell_2 - \ell_1} \tag{D.1}$$

Equation D.1 is Ohm's law. A more common form is $I = (V_2 - V_1)/R$, where the resistance $R = (\ell_2 - \ell_1)/\sigma A$.

Ohm's law and Darcy's law are exactly analogous mathematically. They are experimental laws, they relate a flow to a potential, and they contain a constant of proportionality that is a material property.

The use of capacitors in an electrical analog model permits simulation of time dependent problems. A capacitor stores or releases charge when the voltage across its plates changes. Capacitance is defined as the amount of charge required to raise the voltage across the plates one unit. It is equivalent to the specific storage of an aquifer. Electrical analog models and their relationship to finite difference models are described in detail by Rushton and Redshaw (1979) and by Bennett (1976).

D.3 HEAT FLOW ANALOGY

C. V. Theis (1935) developed an equation for the time-dependent radial flow to a well by translating the heat flow analogy into groundwater terms. Heat flows by conduction in solids from regions of higher temperature to regions of lower temperature. The heat flow is proportional to the temperature gradient

$$q = -K \frac{dT}{d\ell} \tag{D.2}$$

where q is the quantity of heat crossing a unit area in unit time, T is temperature, ℓ is distance, and K is the thermal conductivity. Equation D.2 is Fourier's law. Heat flow is analogous to specific discharge, temperature is analogous to head,

and thermal conductivity is analogous to hydraulic conductivity. Both conductivities are properties of the medium.

The island recharge problem of Section 3.4 can be translated into the following heat flow problem. Consider a glass warming plate whose length is twice its width and whose thickness is b. If the boundaries are kept at 0°C, what heating rate (calories per square centimeter per second) is required to keep the center of the tray at 20°C? Now, suppose the heating element is turned off. The tray continues to stay warm, cooling down only gradually, for the same reason that well levels gradually drop during a drought. The glass stores heat as the temperature rises, and it releases heat as the temperature falls. The measure of this property is the heat capacity, which is defined as the heat required to raise a unit volume a unit degree of temperature.

Carslaw and Jaeger (1959) is the standard treatise on problems of heat conduction. The analytical solutions to the island recharge problem and the reservoir lowering problem in Chapter 4 are translated from Carslaw and Jaeger. An older but very readable text on problems of heat conduction is Ingersoll, Zobel, and Ingersoll (1954).

D.4 STRUCTURAL MECHANICS ANALOGY

The finite element method has its origin in problems of structural mechanics. The groundwater hydrologist can borrow from the extensive literature, including textbooks and computer programs, on establishing the mathematical equivalence between stress-strain analysis and groundwater flow. The computer programs are generally applicable to solving groundwater problems involving Laplace's equation or Poisson's equation. The position of a bending beam is analogous to the position of a water table. The reader is referred to Zienkiewicz (1977) or Cook (1974) as examples of finite element texts based on stress-strain problems.

The correspondence between groundwater variables and their solid mechanics counterparts is more complicated and artificial than in the electrical and heat conduction analogies. The basic quantities dealt with in the mechanics of solids are stress σ and strain ε. Both stress and strain are second-rank tensors that can be expressed as column matrices. They are related by Hooke's law

$$\{\sigma\} = [C]\{\varepsilon\} \tag{D.3}$$

where $[C]$ is a fourth-rank tensor of elastic moduli.

Hooke's law plays the role of Darcy's law in the analogy if the elastic modulus tensor is interpreted properly. Some care needs to be exercised in setting up the analogy between hydraulic conductivity and the elastic moduli. Only two elastic moduli are independent for an isotropic medium. Therefore, if Poisson's ratio v is set to zero and if Young's modulus E is set to hydraulic conductivity K, then it may be necessary to force shear modulus G to equal E in the stress-strain program. Also, the $[C]$ matrix in Table D.2 is for the plane stress case. It has a different form for the plane strain case.

In the mechanics problem, the unknown variables at each point in the problem domain are the horizontal component $u(x, y)$ and the vertical component $v(x, y)$ of the displacement. By fixing v to be zero at all nodal points, the horizontal displacement u is equivalent to head h. In two dimensions, there are three components of strain and stress (Table D.2). The components $\varepsilon_{xx} = \partial u / \partial x$ and $\varepsilon_{yy} = \partial v / \partial y$ are longitudinal strains, and $\varepsilon_{xy} = \partial u / \partial y + \partial v / \partial x$ is the shear strain. Similarly, σ_{xx} and σ_{yy} are normal stresses and τ_{xy} is the shear stress. The strain and stress are equivalent to **grad** h and **q**, respectively, if the second component of the strain and stress are ignored.

In the matrix equation that represents the finite element solution of the stress-strain problem, the assembled global stiffness matrix $[S]$ is analogous to the conductance matrix $[G]$, and the load matrix $\{F\}$ is analogous to the recharge matrix $\{B\}$. The load matrix can handle either point loads (that is, wells) or distributed loads (that is, recharge).

Another analogy is expressed in Equation D.3. The elastic modulus tensor is exactly analogous in its properties to the dispersivity tensor (Bear, 1961). The dispersivity tensor multiplied by the second-rank velocity tensor yields the second-rank dispersion coefficient tensor.