

ANOSTROPIA Y TENSORES

CI51J Hidráulica de Aguas Subterráneas y su Aprovechamiento

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Wang and Anderson. Introduction to Groundwater Modelling. 1982.

APPENDIX **A**

Anisotropy and Tensors

A.1 INTRODUCTION

When the physical properties of a medium are dependent on direction, the medium is said to be anisotropic. Darcy's law was given in Chapter 1 for the case of an isotropic medium. In this appendix, Darcy's law is generalized to a two-dimensional anisotropic medium. The mathematical description of the hydraulic conductivity as a second-rank tensor applies also to the dispersion coefficient as defined in Chapter 8.

A.2 HYDRAULIC CONDUCTIVITY TENSOR

Geologically, Figure A.1(a) represents a cross section of tilted shale beds. The coordinate system has the x axis horizontal and the y axis vertical. We call this

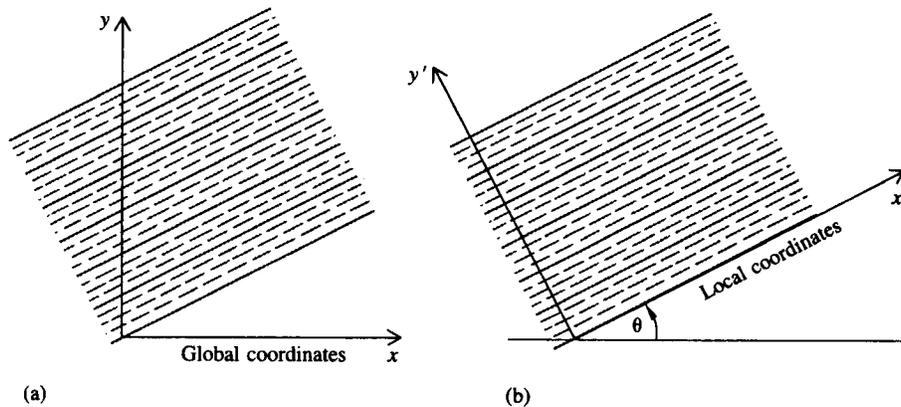


Figure A.1
Tilted shale beds in (a) a global coordinate system and (b) a local coordinate system aligned with the bedding.

the global coordinate system. Suppose that groundwater can flow only parallel to the bedding, that is, the shale is impermeable perpendicular to bedding. In the global coordinate system, a head gradient in the y direction will induce some flow along the beds, and hence there will be an x component of flow. The relationship between q_x and $\partial h/\partial y$ is not covered by the form of Darcy's law introduced in Chapter 1. The generalization requires that each component of the vector \mathbf{q} be linearly proportional to each component of the vector $\mathbf{grad} h$.

$$q_x = -K_{11} \frac{\partial h}{\partial x} - K_{12} \frac{\partial h}{\partial y} \quad (\text{A.1a})$$

$$q_y = -K_{21} \frac{\partial h}{\partial x} - K_{22} \frac{\partial h}{\partial y} \quad (\text{A.1b})$$

The ratio q_y/q_x is not equal to the ratio $(\partial h/\partial y)/(\partial h/\partial x)$; that is, \mathbf{q} does not point in the same direction as $\mathbf{grad} h$.

Instead of a single proportionality constant K as in Equation 1.7, there are now four proportionality constants. The hydraulic conductivity is represented in matrix form as

$$[K] = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \quad (\text{A.2})$$

Equation A.2 is the tensor representation of hydraulic conductivity for an anisotropic medium. Hydraulic conductivity is a second-rank tensor because it relates two vectors that are first-rank tensors. A scalar quantity is a zeroth-rank tensor.

Suppose a coordinate system more natural to the geological situation were chosen. The coordinate system with the x' direction parallel to bedding and the y' direction perpendicular to bedding will be called the local coordinate system (Figure A.1b). The prime notation will be used to distinguish these coordinates from the global coordinates. In the local coordinate system, the generalized form of Darcy's law is

$$q_{x'} = -K'_{11} \frac{\partial h}{\partial x'} - K'_{12} \frac{\partial h}{\partial y'} \quad (\text{A.3a})$$

$$q_{y'} = -K'_{21} \frac{\partial h}{\partial x'} - K'_{22} \frac{\partial h}{\partial y'} \quad (\text{A.3b})$$

In the local coordinate system, a gradient in the y' direction will not produce flow in the x' direction, that is, $K'_{12} = 0$. Similarly, a gradient in the x' direction will not produce flow in the y' direction, and $K'_{21} = 0$. The components of the hydraulic conductivity tensor depend on the choice of coordinate system. In the coordinate system in which the off-diagonal components of the tensor are zero, the coordinate axes directions are called the principal directions.

A.3 COORDINATE SYSTEM ROTATION

Except for choice of coordinate system, the hydrogeologic problems represented in Figures A.1a and A.1b are the same. There must then be a mathematical description that transforms the problem from one set of coordinates to the other, so that the simulation of the physical situation remains unchanged. The magnitudes and directions of the two vectors, specific discharge q and gradient of head $\text{grad } h$, must be the same in the two coordinate systems. Only their

representation in terms of components will be different in the different coordinate systems. Similarly, the components of the hydraulic conductivity tensor will be different in the two coordinate systems, but the same proportionality between \mathbf{q} and $\mathbf{grad} h$ must be maintained. We use these constraints to show the mathematical transformation of the hydraulic conductivity tensor for a rotation in two dimensions.

Let θ be the counterclockwise rotation angle of the local coordinate system relative to the global coordinate system. The components of a vector \mathbf{q} in the global coordinate system are related to those in the local coordinate system through the matrix equation

$$\begin{pmatrix} q_{x'} \\ q_{y'} \end{pmatrix} = [R] \begin{pmatrix} q_x \\ q_y \end{pmatrix} \quad (\text{A.4})$$

where the rotation matrix $[R]$ is defined by

$$[R] = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad (\text{A.5})$$

The rotation matrix is described in most texts on vector algebra. Just as in Equation A.4, the components of the gradient vector in the global coordinate system are related to those in the local coordinate system through the matrix equation

$$\begin{pmatrix} \frac{\partial h}{\partial x'} \\ \frac{\partial h}{\partial y'} \end{pmatrix} = [R] \begin{pmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \end{pmatrix} \quad (\text{A.6})$$

Equation A.3, Darcy's law, can be written in matrix form also.

$$\begin{pmatrix} q_{x'} \\ q_{y'} \end{pmatrix} = -[K'] \begin{pmatrix} \frac{\partial h}{\partial x'} \\ \frac{\partial h}{\partial y'} \end{pmatrix} \quad (\text{A.7})$$

Substituting Equations A.4 and A.6 into Equation A.7 and multiplying across by the inverse of $[R]$ gives

$$\begin{pmatrix} q_x \\ q_y \end{pmatrix} = -[R]^{-1}[K'][R] \begin{pmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \end{pmatrix} \quad (\text{A.8})$$

The inverse rotation matrix $[R]^{-1}$ is obtained by substituting $-\theta$ for θ . It rotates a vector from the local coordinate system back into the global system. The $[R]^{-1}$ matrix is

$$[R]^{-1} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (\text{A.9})$$

Equation A.8 is the matrix form of Darcy's law in global coordinates when we make the identification

$$[K] = [R]^{-1}[K'][R] \quad (\text{A.10})$$

Equation A.10 is the basic result of this appendix. It describes how the hydraulic conductivity tensor components transform with a coordinate rotation.

In most flow problems, it is possible to define the global coordinate system to coincide with the principal directions of the hydraulic conductivity tensor. In this case, one need only define K_{11} and K_{22} , because K_{12} and K_{21} will be zero. If it is not possible to define such a global coordinate system, it will be necessary to work with both local and global coordinate systems. Equation A.10 is used to find K_{11} , K_{22} , K_{12} , and K_{21} for each node or element, given that in the local coordinate system $K'_{12} = K'_{21} = 0$, and K'_{11} and K'_{22} are supplied from field data.

In contaminant transport problems where the flow field is not uniform, the global and local coordinate systems do not coincide in general. In this case, the dispersion coefficient tensor must be rotated between the coordinate systems as necessary throughout the problem domain according to Equation 8.16.