tinuation of $\log f(z)$ in a neighborhood of the point. Then e^g gives the analytic continuation of f to $U \cup \gamma$. By Proposition 1.3 (in its isomorphic form corresponding to the analytic arc γ), we conclude that the analytic extension of f is injective on a sufficiently small open set containing $U \cup \gamma$, thus concluding the proof.

IX, §2. EXERCISES

1. Let C be an arc of unit circle $|z| = 1$, and let U be an open set inside the circle, having that arc as a piece of its boundary. If f is analytic on U if f maps U into the upper half plane, f is continuous on C, and takes real values on C, show that f can be continued across C by the relation

$$f(z) = \overline{f(1/\bar{z})}.$$

2. Suppose, on the other hand, that instead of taking real values on C, f takes on values on the unit circle, that is,

$$|f(z)| = 1 \quad \text{for} \quad z \text{ on } C.$$

Show that the analytic continuation of f across C is now given by

$$f(z) = 1/\overline{f(1/\bar{z})}.$$

3. Let f be a function which is continuous on the closed unit disc and analytic on the open disc. Assume that $|f(z)| = 1$ whenever $|z| = 1$. Show that the function f can be extended to a meromorphic function, with at most a finite number of poles in the whole plane.

4. Let f be a meromorphic function on the open unit disc and assume that f has a continuous extension to the boundary circle. Assume also that f has only a finite number of poles in the unit disc, and that $|f(z)| = 1$ whenever $|z| = 1$. Prove that f is a rational function.

5. Work out the exercise left for you in the text, that is:

Let W be an open neighborhood of a real interval $[a, b]$. Let g be analytic on W, and assume that $g'(t) \neq 0$ for all $t \in [a, b]$, and g is injective on $[a, b]$. Then there exists an open subset W_0 of W containing $[a, b]$ such that g is an analytic isomorphism of W_0 with its image.

[*Hint*: First, by compactness, show that there is some neighborhood of $[a, b]$ on which g' does not vanish, and so g is a local isomorphism at each point of this neighborhood. Let $\{W_n\}$ be a sequence of open sets shrinking to $[a, b]$, for instance the set of points at distance $< 1/n$ from $[a, b]$. Suppose g is not injective on each W_n. Let $z_n \neq z_n'$ be two points in W_n such that $g(z_n) = g(z_n')$. The sequences $\{z_n\}$ and $\{z_n'\}$ have convergent subsequences, to points on $[a, b]$. If these limit points are distinct, this contradicts the injectivity of g on the real