that if

$$F(z) = \frac{az + b}{cz + d} \quad \text{and} \quad G(z) = \frac{a'z + b'}{c'z + d'}$$

then there exists a complex number λ such that

$$a' = \lambda a, \quad b' = \lambda b, \quad c' = \lambda c, \quad d' = \lambda d.$$

Thus the matrices representing F and G differ by a scalar.

6. Consider the fractional linear map

$$F(z) = \frac{z - i}{z + i}.$$

What is the image of the real line $\mathbf{R}$ under this map? (You have encountered this map as an isomorphism between the upper half plane and the unit disc.)

7. Let F be the fractional linear map $F(z) = (z - 1)/(z + 1)$. What is the image of the real line under this map? (Cf. Example 9 of §4.)

8. Let $F(z) = z/(z - 1)$ and $G(z) = 1/(1 - z)$. Show that the set of all possible fractional linear maps which can be obtained by composing F and G above repeatedly with each other in all possible orders in fact has six elements, and give a formula for each one of these elements. [*Hint*: Compute F^2, F^3, G^2, G^3, $F \circ G$, $G \circ F$, etc.]

9. Let $F(z) = (z - i)/(z + i)$. What is the image under F of the following sets of points:
   (a) The upper half line it, with $t \geqq 0$.
   (b) The circle of center 1 and radius 1.
   (c) The horizontal line $i + t$, with $t \in \mathbf{R}$.
   (d) The half circle $|z| = 2$ with $\operatorname{Im} z \geqq 0$.
   (e) The vertical line $\operatorname{Re} z = 1$ and $\operatorname{Im} z \geqq 0$.

10. Find fractional linear maps which map:
    (a) $0, 1, 2$ to $1, 0, \infty$
    (b) $i, -1, 1$ to $1, 0, \infty$
    (c) $0, 1, 2$ to $i, -1, 1$

11. Let $F(z) = (z + 1)/(z - 1)$. Describe the image of the line $\operatorname{Re}(z) = c$ for a real number c. (Distinguish $c = 1$ and $c \neq 1$. In the second case, the image is a circle. Give its center and radius.)

12. Let z_1, z_2, z_3, z_4 be distinct complex numbers. Define their **cross ratio** to be

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}.$$

   (a) Let F be a fractional linear map. Let $z_i' = F(z_i)$ for $i = 1, \ldots, 4$. Show that the cross ratio of z_1', z_2', z_3', z_4' is the same as the cross ratio of