estimate reads

$$|f^{(n)}(z)| \leqq \frac{n!\,2^{n+1}}{R^n}\|f\|_R,$$

which is thus entirely in terms of f, n, and R.

Finally we return to reconsider Theorem 3.2 in light of the fact that a holomorphic function is analytic.

Theorem 7.9 (Morera's Theorem). *Let U be an open set in $\mathbf{C}$ and let f be continuous on U. Assume that the integral of f along the boundary of every closed rectangle contained in U is 0. Then f is holomorphic.*

Proof. By Theorem 3.2, we know that f has a local primitive g at every point on U, and hence that g is holomorphic. By Theorem 7.2, we conclude that g is analytic, and hence that $g' = f$ is analytic, as was to be shown.

We have now come to the end of a chain of ideas linking complex differentiability and power series expansions. The next two chapters treat different applications, and can be read in any order, but we have to project the book in a totally ordered way on the page axis, so we have to choose an order for them. The next chapter will study more systematically a global version of Cauchy's formula and winding numbers, which amounts to studying the relation between an integral and the winding number which we already encountered in some way via the logarithm. After that in Chapter V, we return to analytic considerations and estimates.

III, §7. EXERCISES

1. Find the integrals over the unit circle γ:

(a) $\displaystyle\int_\gamma \frac{\cos z}{z}\,dz$ (b) $\displaystyle\int_\gamma \frac{\sin z}{z}\,dz$ (c) $\displaystyle\int_\gamma \frac{\cos(z^2)}{z}\,dz$

2. Write out completely the proof of Theorem 7.6 to see that all the steps in the proof of Theorem 7.3 apply.

3. Prove Corollary 7.4.