Replace z by $2\pi iz$ to show that

$$\pi z \cot \pi z = \sum_{n=0}^{\infty} (-1)^n \frac{(2\pi)^{2n}}{(2n)!} B_{2n} z^{2n}.$$

5. Express the power series for $\tan z$, $z/\sin z$, $z \cot z$, in terms of Bernoulli numbers.

6. **(Difference Equations).** Given complex numbers a_0, a_1, u_1, u_2 define a_n for $n \geqq 2$ by

$$a_n = u_1 a_{n-1} + u_2 a_{n-2}.$$

If we have a factorization

$$T^2 - u_1 T - u_2 = (T - \alpha)(T - \alpha'), \qquad \text{and} \quad \alpha \neq \alpha',$$

show that the numbers a_n are given by

$$a_n = A\alpha^n + B\alpha'^n$$

with suitable A, B. Find A, B in terms of a_0, a_1, α, α'. Consider the power series

$$F(T) = \sum_{n=0}^{\infty} a_n T^n.$$

Show that it represents a rational function, and give its partial fraction decomposition.

7. More generally, let $a_0, \ldots, a_{r-1}$ be given complex numbers. Let $u_1, \ldots, u_r$ be complex number such that the polynomial

$$P(T) = T^r - (u_1 T^{r-1} + \cdots + u_r)$$

has distinct roots $\alpha_1, \ldots, \alpha_r$. Define a_n for $n \geqq r$ by

$$a_n = u_1 a_{n-1} + \cdots + u_r a_{n-r}.$$

Show that there exist numbers $A_1, \ldots, A_r$ such that for all n,

$$a_n = A_1 \alpha_1^n + \cdots + A_r \alpha_r^n.$$

II, §2. CONVERGENT POWER SERIES

We first recall some terminology about series of complex numbers. Let $\{z_n\}$ be a sequence of complex numbers. Consider the series

$$\sum_{n=1}^{\infty} z_n.$$