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PROYECTOS DE INGENIERIA

$$D_2 = \left(\frac{\partial f}{\partial p} + x\frac{\partial f}{\partial r}\right)\left[\frac{\partial^2 f}{\partial h_1 \partial h_3} + 2gg'\frac{\partial^2 f}{\partial h_2 \partial h_3}\left(2x\frac{\partial f}{\partial u} + \frac{\partial f}{\partial w}\right) + \frac{\partial^2 f}{\partial h_3^2}\left(y\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\right)\right] +$$

$$\frac{\partial f}{\partial h_3}\cdot\left[\left(y\frac{\partial^2 f}{\partial r\partial p} + \frac{\partial^2 f}{\partial s \partial p}\right) + \frac{\partial f}{\partial r} + x\left(y\frac{\partial^2 f}{\partial r^2} + \frac{\partial^2 f}{\partial s\partial r}\right)\right]$$

Luego, sumando D_1 y D_2 se obtiene:

$$\frac{\partial^2 F}{\partial x \partial y} = \left\{\left[2(g')^2 + 2gg''\right]\cdot\frac{\partial f}{\partial h_2}\left(2x\frac{\partial f}{\partial u} + \frac{\partial f}{\partial w}\right)\left(2y\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}\right) + 2gg'\left(2y\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}\right)\left[\frac{\partial^2 f}{\partial h_1 \partial h_2} + \right.\right.$$

$$2gg'\frac{\partial^2 f}{\partial h_2^2}\left(2x\frac{\partial f}{\partial u} + \frac{\partial f}{\partial w}\right) + \frac{\partial^2 f}{\partial h_3 \partial h_2}\left(y\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\right)\bigg] + 2gg'\frac{\partial f}{\partial h_2}\left[2y\left(2x\frac{\partial^2 f}{\partial u \partial v} + \frac{\partial^2 f}{\partial w \partial v}\right) + \right.$$

$$+\left(2x\frac{\partial^2 f}{\partial u \partial w} + \frac{\partial^2 f}{\partial w^2}\right)\bigg]\bigg\} + \left\{\left(\frac{\partial f}{\partial p} + x\frac{\partial f}{\partial r}\right)\left[\frac{\partial^2 f}{\partial h_1 \partial h_3} + 2gg'\frac{\partial^2 f}{\partial h_2 \partial h_3}\left(2x\frac{\partial f}{\partial u} + \frac{\partial f}{\partial w}\right) + \frac{\partial^2 f}{\partial h_3^2}\left(y\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\right)\right]\right.$$

$$\left. + \frac{\partial f}{\partial h_3}\left[\left(y\frac{\partial^2 f}{\partial r \partial p} + \frac{\partial^2 f}{\partial s \partial p}\right) + \frac{\partial f}{\partial r} + x\left(y\frac{\partial^2 f}{\partial r^2} + \frac{\partial^2 f}{\partial s \partial r}\right)\right]\right\} \qquad (8) \quad \blacksquare$$

Ahora evaluemos esta expresión y calculemos directamente para comprobar.

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)\bigg|_{(5)} = -2(x^2 - y - xy) - 2(1+x)(2x-y) - 4e^{4xy} - 16xy\,e^{4xy} \qquad (9)$$

Para evaluar (8) es necesario obtener las derivadas que allí aparecen.

$$\frac{\partial^2 f}{\partial h_1 \partial h_2} = 0 \;;\; \frac{\partial^2 f}{\partial h_2^2} = 0 \;;\; \frac{\partial^2 f}{\partial h_3 \partial h_2} = 0 \;;\; \frac{\partial^2 f}{\partial u \partial v} = 0 \;;\; \frac{\partial^2 f}{\partial w \partial v} = 0 \;;\; \frac{\partial^2 f}{\partial u \partial w} = 0 \;;\; \frac{\partial^2 f}{\partial w^2} = 2$$

$$\frac{\partial^2 f}{\partial h_1 \partial h_3} = 0 \;;\; \frac{\partial^2 f}{\partial h_2 \partial h_3} = 0 \;;\; \frac{\partial^2 f}{\partial h_3^2} = 2 \;;\; \frac{\partial^2 f}{\partial r \partial p} = 0 \;;\; \frac{\partial^2 f}{\partial s \partial p} = 0 \;;\; \frac{\partial^2 f}{\partial r^2} = 0 \;;\; \frac{\partial^2 f}{\partial s \partial r} = 0$$

Reemplazando:

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)\bigg|_{(8)} = \left\{\left[2\cdot(e^f)^2 + 2e^f\cdot e^f\right]\cdot(-1)\left(2x(-1) + 2w\right)\left(2y(-1) + 2w\right) + 2(e^f)^2\left(2y(-1) + 2w\right)\left[0 + 0 + 0\right]\right.$$

$$+ 2(e^f)^2\cdot(-1)\left[2y\cdot 0 + \quad\right]\Big\} + \left\{\left((-1) + x(-1)\right)\left[0 + 0 + 2(y(-1) + 2s\right] + 2h_3\left[0 + (-1) + x\cdot 0\right]\right.$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)\bigg|_{(8)} = \left\{-4e^{f\cdot 2}\left(2x(-1) + 2(x+y)\right)\left(-2y + 2(x+y)\right) - 4e^{2f}\right\} - (1+x)\left[2(-y + 2x)\right] + 2(x^2 - y - xy)(-1)$$

$$= -4e^{4xy}\cdot(2y)(2x) - 4e^{4xy} - 2(1+x)(2x-y) - 2(x^2 - y - xy) \qquad (10)$$

Al comparar las expresiones (9) y (10) se comprueba la validez de (8)

P3) Sean $f, g \in C^2(\mathbb{R})$ con $f: \mathbb{R}^3 \to \mathbb{R}$; $g: \mathbb{R} \to \mathbb{R}$
Se define

$$F(x,y) = f(x,\, g^2(f(x^2, y^2, x+y)),\, f(y, xy, x))$$

a) Se pide $\dfrac{\partial F}{\partial x}$, $\dfrac{\partial F}{\partial y}$, $\dfrac{\partial^2 F}{\partial x \partial y}$

b) Comprobación usando $g(t) = e^t$ $\wedge$ $f(x,y,z) = z^2 - (x+y)$

Solución

Se definen algunas funciones y variables auxiliares

$F(x,y) = f(h_1, h_2, h_3)$ donde $h_1(x) = x$

$h_2(g) = g^2(f(u, v, w))$ $u = x^2$; $v = y^2$; $w = x+y$

$\vec{H} = (h_1, h_2, h_3)$; $\vec{U} = (u, v, w)$ $h_3(f) = f(p, r, s)$ $p = y$; $r = xy$; $s = x$

*) Cálculo de $\dfrac{\partial F}{\partial x}$

$$\frac{\partial F}{\partial x} = \left.\frac{\partial f}{\partial h_1}\right|_H \cdot \frac{\partial h_1}{\partial x} + \frac{\partial f}{\partial h_2} \cdot \frac{\partial h_2}{\partial g} \cdot \frac{\partial g}{\partial f}\left(\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x}\right) + \frac{\partial f}{\partial h_3} \cdot \frac{\partial h_3}{\partial f}\left(\frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial x}\right)$$

Pero $\dfrac{\partial h_2}{\partial g} = 2g(t)$; $\dfrac{\partial u}{\partial x} = 2x$; $\dfrac{\partial v}{\partial x} = 0$; $\dfrac{\partial w}{\partial x} = 1$; $\dfrac{\partial r}{\partial x} = y$; $\dfrac{\partial s}{\partial x} = 1$; $\dfrac{\partial h_1}{\partial x} = 1$

Además nótese que $\left.\dfrac{\partial h_3}{\partial f}\right|_f = 1$ y que $\left.\dfrac{\partial h_3}{\partial f}\right|_f \neq \left.\dfrac{\partial f}{\partial h_3}\right|_{\vec{H}}$; $\dfrac{\partial h_2}{\partial g} = 2g$; $\dfrac{\partial g}{\partial f} = g'(t)$

$$\therefore \frac{\partial F}{\partial x} = \frac{\partial f}{\partial h_1} + 2gg'\frac{\partial f}{\partial h_2}\left(2x\frac{\partial f}{\partial u} + \frac{\partial f}{\partial w}\right) + \frac{\partial f}{\partial h_3}\left(y\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\right) \qquad (1)$$

*) Cálculo de $\partial F / \partial y$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial h_1} \cdot \frac{\partial h_1}{\partial y} + \frac{\partial f}{\partial h_2} \cdot \frac{\partial h_2}{\partial g} \cdot \frac{\partial g}{\partial f}\left(\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial y}\right) + \frac{\partial f}{\partial h_3} \cdot \frac{\partial h_3}{\partial f}\left(\frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial y}\right)$$

$$\frac{\partial F}{\partial y} = 2gg'\frac{\partial f}{\partial h_2}\left(2y\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}\right) + \frac{\partial f}{\partial h_3}\left(\frac{\partial f}{\partial p} + x\frac{\partial f}{\partial r}\right) \qquad (2)$$

comprobación de $\partial F/\partial x \wedge \partial F/\partial y$

dada $f(x,y,z) = z^2 - (x+y) \Rightarrow f(h_1,h_2,h_3) = h_3^2 - (h_1+h_2)$

$$f(u,v,w) = w^2 - (u+v)$$
$$f(p,r,s) = s^2 - (p+r)$$

$$\therefore f(x^2,y^2,x+y) = (x+y)^2 - (x^2+y^2) = 2xy \qquad g(t) = e^t$$
$$f(y,xy,x) = x^2 - (y+xy)$$

$$F(x,y) = f(x, g^2(2xy), x^2-y-xy)$$
$$= (x^2-y-xy)^2 - x - (e^{2xy})^2$$

$$F(x,y) = (x^2-y-xy)^2 - x - e^{4xy} \qquad (3)$$

*) Cálculo directo : se deriva (3) con respecto a x e y

$$\left.\frac{\partial F}{\partial x}\right|_{directo} = 2(x^2-y-xy)(2x-y) - 1 - 4ye^{4xy} \qquad (4)$$

$$\left.\frac{\partial F}{\partial y}\right|_{directo} = -2(x^2-y-xy)(1+x) - 4xe^{4xy} \qquad (5)$$

*) Cálculo usando las expresiones (1) y (2)

$$\frac{\partial f}{\partial h_1} = -1 \;;\; \frac{\partial f}{\partial h_2} = -1 \;;\; \frac{\partial g}{\partial t} = e^t \;;\; \frac{\partial f}{\partial u} = -1 \;;\; \frac{\partial f}{\partial w} = 2w \;;\; \frac{\partial f}{\partial h_3} = 2h_3 \;;\; \frac{\partial f}{\partial r} = -1 \;;\; \frac{\partial f}{\partial s} = 2s$$

Reemplazando en (1)

$$\left.\frac{\partial F}{\partial x}\right|_{(1)} = -1 + 2e^t \cdot e^t \cdot (-1)\left(2x(-1) + 2w\right) + 2h_3\left(y(-1) + 2s\right)$$
$$= -1 - 2e^{4xy} \cdot \left[-\cancel{2x} + 2(\cancel{x}+y)\right] + 2(x^2-y-xy)(2x-y)$$

$$\left.\frac{\partial F}{\partial x}\right|_{(1)} = 2(x^2-xy-y)(2x-y) - 1 - 4ye^{4xy} \qquad (6)$$

Al comparar (4) y (6) se comprueba la validez de (1)

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Ahora, evaluando la fórmula (2)

$$\left.\frac{\partial F}{\partial y}\right|_{(2)} = 2\cdot e^{f}\cdot e^{f}(-1)\Big(2y(-1)+2w\Big) + 2h_3\big((-1)+x\cdot(-1)\big)$$

$$= -2e^{4xy}\big(-2y+2(x+y)\big) + 2(x^2-y-xy)(-1-x)$$

$$\left.\frac{\partial F}{\partial y}\right|_{(2)} = -4xe^{4xy} - 2(x^2-y-xy)(1+x) \qquad (7)$$

Al comparar (5) y (7) se comprueba la validez de la expresión (2).

*) Cálculo de $\dfrac{\partial^2 F}{\partial x\partial y}$

$$\frac{\partial^2 F}{\partial x\partial y} = \frac{\partial}{\partial x}\left(2gg'\frac{\partial f}{\partial h_2}\left(2y\frac{\partial f}{\partial v}+\frac{\partial f}{\partial w}\right)\right) + \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial h_3}\cdot\left(\frac{\partial f}{\partial p}+x\frac{\partial f}{\partial r}\right)\right]$$

Calculando cada derivada por separado.

$$D_1 = 2\frac{\partial g}{\partial x}\cdot g'\frac{\partial f}{\partial h_2}\left(2y\frac{\partial f}{\partial v}+\frac{\partial f}{\partial w}\right) + 2\frac{\partial g'}{\partial x}\cdot g\frac{\partial f}{\partial h_2}\left(2y\cdot\frac{\partial f}{\partial v}+\frac{\partial f}{\partial w}\right) + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial h_2}\right)\cdot 2gg'\left(2y\frac{\partial f}{\partial v}+\frac{\partial f}{\partial w}\right)$$
$$+\left[2y\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}\right)+\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial w}\right)\right]\left(2gg'\left(\frac{\partial f}{\partial h_2}\right)\right)$$

Pero

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial f}\cdot\left(\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x}\right) = g'\cdot\left(2x\cdot\frac{\partial f}{\partial u}+\frac{\partial f}{\partial w}\right)$$

$$\frac{\partial g'}{\partial x} = \frac{\partial g'}{\partial f}\cdot\left(\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x}\right) = g''\cdot\left(2x\cdot\frac{\partial f}{\partial u}+\frac{\partial f}{\partial w}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial h_2}\right) = \frac{\partial^2 f}{\partial h_1\partial h_2}\cdot\frac{\partial h_1}{\partial x} + \frac{\partial^2 f}{\partial h_2^2}\cdot\frac{\partial h_2}{\partial g}\cdot\frac{\partial g}{\partial f}\cdot\left(\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x}\right) + \frac{\partial^2 f}{\partial h_3\partial h_2}\cdot\frac{\partial h_3}{\partial f}\cdot\left(\frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial h_2}\right) = \frac{\partial^2 f}{\partial h_1\partial h_2} + \frac{\partial^2 f}{\partial h_2^2}\cdot 2gg'\left(2x\frac{\partial f}{\partial u}+\frac{\partial f}{\partial w}\right) + \frac{\partial^2 f}{\partial h_3\partial h_2}\left(y\frac{\partial f}{\partial r}+\frac{\partial f}{\partial s}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}\right) = \frac{\partial^2 f}{\partial u\partial v}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial v^2}\cdot\frac{\partial v}{\partial x} + \frac{\partial^2 f}{\partial w\partial v}\cdot\frac{\partial w}{\partial x} = \frac{\partial^2 f}{\partial u\partial v}\cdot 2x + \frac{\partial^2 f}{\partial w\partial v}$$

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$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial w}\right) = \frac{\partial^2 f}{\partial u \partial w}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial v \partial w}\cdot\frac{\partial v}{\partial x} + \frac{\partial^2 f}{\partial w^2}\cdot\frac{\partial w}{\partial x} = 2x\cdot\frac{\partial^2 f}{\partial u \partial w} + \frac{\partial^2 f}{\partial w^2}$$

Reemplazando en la expresión de D_1 queda:

$$D_1 = \left[2\cdot(g')^2\cdot\frac{\partial f}{\partial h_2}\cdot\left(2x\frac{\partial f}{\partial u} + \frac{\partial f}{\partial w}\right)\left(2y\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}\right)\right] + \left[2g''\cdot g\frac{\partial f}{\partial h_2}\left(2y\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}\right)\left(2x\frac{\partial f}{\partial u} + \frac{\partial f}{\partial w}\right)\right] +$$

$$+ 2gg'\left(2y\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}\right)\left[\frac{\partial^2 f}{\partial h_1 \partial h_2} + 2gg'\frac{\partial^2 f}{\partial h_2^2}\left(2x\frac{\partial f}{\partial u} + \frac{\partial f}{\partial w}\right) + \frac{\partial^2 f}{\partial h_3 \partial h_2}\left(y\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\right)\right] +$$

$$+\left(2gg'\frac{\partial f}{\partial h_2}\right)\left[2y\left(2x\frac{\partial^2 f}{\partial u \partial v} + \frac{\partial^2 f}{\partial w \partial v}\right) + \left(2x\frac{\partial^2 f}{\partial u \partial w} + \frac{\partial^2 f}{\partial w^2}\right)\right]$$

Ahora, se calculará la otra derivada (D_2)

$$D_2 = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial h_3}\left(\frac{\partial f}{\partial p} + x\frac{\partial f}{\partial r}\right)\right]$$

$$D_2 = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial h_3}\right)\cdot\left(\frac{\partial f}{\partial p} + x\frac{\partial f}{\partial r}\right) + \frac{\partial f}{\partial h_3}\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial p}\right) + \frac{\partial f}{\partial r} + x\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial r}\right)\right]$$

Pero

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial h_3}\right) = \frac{\partial^2 f}{\partial h_1 \partial h_3}\cdot\frac{\partial h_1}{\partial x} + \frac{\partial^2 f}{\partial h_2 \partial h_3}\cdot\frac{\partial h_2}{\partial g}\cdot\frac{\partial g}{\partial f}\left(\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x}\right) + \frac{\partial^2 f}{\partial h_3^2}\,\frac{\partial h_3}{\partial f}\cdot\left(\frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial h_3}\right) = \frac{\partial^2 f}{\partial h_1 \partial h_3} + 2gg'\frac{\partial^2 f}{\partial h_2 \partial h_3}\left(2x\frac{\partial f}{\partial u} + \frac{\partial f}{\partial w}\right) + \frac{\partial^2 f}{\partial h_3^2}\cdot\left(y\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial p}\right) = \frac{\partial^2 f}{\partial r \partial p}\cdot\frac{\partial r}{\partial x} + \frac{\partial^2 f}{\partial p^2}\cdot\frac{\partial p}{\partial x} + \frac{\partial^2 f}{\partial s \partial p}\cdot\frac{\partial s}{\partial x} = y\frac{\partial^2 f}{\partial r \partial p} + \frac{\partial^2 f}{\partial s \partial p}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial r}\right) = \frac{\partial^2 f}{\partial r^2}\cdot\frac{\partial r}{\partial x} + \frac{\partial^2 f}{\partial p \partial r}\cdot\frac{\partial p}{\partial x} + \frac{\partial^2 f}{\partial s \partial r}\cdot\frac{\partial s}{\partial x} = y\frac{\partial^2 f}{\partial r^2} + \frac{\partial^2 f}{\partial s \partial r}$$

Reemplando en la expresión de D_2