PROBLEM SET #2- ANSWER KEY

Problem 1

Recall the definition of an open set:

> A set $S \subset X$ is said to be open in X w.r.t. a metric d if $\forall s \in S$ $\exists \varepsilon > 0$ such that $B_d(s,\varepsilon) = \{x \in X | d(x,s) < \varepsilon\} \subset S$.

a) Show that the union $S = \cup_{i \in I} S_i$ of arbitrarily many open sets S_i is an open set.

(1) For any element $s \in S$ we know by the definition of an union of sets that it must belong to at least one set S_k, i.e., $\exists k \in I$ with $s \in S_k$

(2) Since the set S_k is open, there is an ε-ball around s that lies totally within the set S_k, i.e., $\exists \varepsilon > 0$ such that $B_d(s,\varepsilon) = \{x \in X | d(x,s) < \varepsilon\} \subset S_k$.

(3) Since any set S_k that is part of the union S is a subset of the union S we know that the ε-ball from (2) also lies within S. In mathematical terms: By the definition of a union: $S_k \subset S$ and hence from (2) $\exists \varepsilon > 0$ such that $B_d(s,\varepsilon) \subset S_k \subset S$.

(4) Hence we can fit an ε-ball around any element in the set and the ball lies within the set. The definition for an open set is thus fullfilled.

b) Show that the intersection of finitely many open sets S_i is an open set. Without loss of generality we can consider sets $S_1, S_2, ..., S_N$ with $N \in \mathbb{N}$ and $S = \cap_{i=1}^{N} S_i$.

(1) For any element $s \in S$ we know by the definition of the intersection of sets that it must belong to *all* sets S_i $i = 1...N$.

(2) Since all sets S_i are open, there is an ε-ball around s that lies totally within the set S_i for all sets S_i, i.e., $\exists \varepsilon_i > 0$ such that $B_d^{(i)}(s,\varepsilon_i) = \{x \in X | d(x,s) < \varepsilon_i\} \subset S_i$ for $i = 1...N$.

(3) Let $\varepsilon = \min\{\varepsilon_i, i = 1...N\} > 0$.

(4) By the construction of ε we know that $B_d(s,\varepsilon) \subset B_d^i(s,\varepsilon_i)$ for $i = 1...N$. (The ball with radius ε is contained in all the balls with the larger radius ε_i).

(5) We therefore know from (4) and (2) that $B_d(s,\varepsilon) \subset S_i$ for all $i = 1...N$ and hence $B_d(s,\varepsilon) \subset S$.

(6) Hence we can fit an ε-ball around any element in the set and the ball lies within the set. The definition for an open set is thus fullfilled.

c) *What step in part (b) no longer holds in general if you consider infinitely many sets?* This would be step (3), i.e., the minimum of infinitely many elements soemtimes doesn't exist.

d) Show that the intersection $S = \cap_{i\in I} S_i$ of arbitrarily many closed sets S_i is a closed set.

(1) By definition a set S_i is closed if its complement is open, i.e., the sets $X \setminus S_i$ is open for $i \in I$.

(2) From part (a) we know that the union of arbitarily many open sets is open and we thus know that $\cup_{i\in I}(X \setminus S_i) = X \setminus (\cap_{i\in I} S_i)$ is open.

(3) Hence the intersection of arbitraily many closed sets $\cap_{i\in I} S_i$ is closed as its complement $X \setminus (\cap_{i\in I} S_i)$ is open after (2).

Problem 2

Recall the definition and theorem from lecture notes #8:

(A) A point $s \in X$ is called an accumulation point of a set $S \subset X$ if $\forall \varepsilon > 0$ the ball $B_d(s,\varepsilon)$ contains a point $s_1 \in S, s_1 \neq s$

(B) A point belongs to the boundary of a set $S \subset X$ iff $\forall \varepsilon > 0, \exists s_2, s_3 \in B_d(s,\varepsilon)$ such that $s_2 \in S$ and $s_3 \in X \setminus S$.

Let's consider the problems

a) *Every accumulation point of a set $S \subset X$ is a boundary point of S?*

False: Consider for example the set $S = (0,1)$ in the universe $X = \mathbb{R}$ and the Pythagorian metric. All points of S are accumulation points, but none of them are boundary points.

b) *Every boundary point of a set $S \subset X$ is an accumulation point of S?*

False: Consider for example the set $S = \{0\}$ in the universe $X = \mathbb{R}$ and the Pythagorian metric. The only element of S is a boundary point but not an accumulation point.

c) *Every accumulation point s of a set $S \subset X$ that is not an element of S is a boundary point of S?*

True: You have to show that if $s \notin S$ than $A \Rightarrow B$.

(1) By assumption we are given that s does *not* belong to the set S, i.e., $s \notin S$

(2) We are given (A) and thus know that $\forall \varepsilon > 0$ the ball $B_d(s, \varepsilon)$ contains a point $s_1 \in S, s_1 \neq s$.

(3) Hence $\forall \varepsilon > 0, \exists s_2 = s_1$ and $s_3 = s \in B_d(s, \varepsilon)$ and by (2) we know that $s_2 \in S$ and by (1) we know that $s_3 \in X \setminus S$. Consequently, s satisfies (B).

d) *Every boundary point s of a set $S \subset X$ that is not an element of S is an accumulation point of S?*

True: You have to show that if $s \notin S$ than $B \Rightarrow A$.

(1) By assumption we are given that s does *not* belong to the set S, i.e., $s \notin S$

(2) We are given (B) and thus know that $\forall \varepsilon > 0$ the ball $B_d(s, \varepsilon)$ contains points $s_2 \in S$ and $s_3 \in X \setminus S$.

(3) Hence, $\forall \varepsilon > 0, \exists s_1 = s_2$ with $s_1 \in B_d(s, \varepsilon)$ by (2) and from (1) we know that $s_1 \neq s$. Consequently, s satifies (A)

Problem 3

Let's choose the following notation

(A) S is closed

(B) $cl(S) = S$

You have to show that $A \Leftrightarrow B$.

"$\Rightarrow$" (1) You are given (A), i.e., the set S is closed.

(2) The closure of S is the intersection of all closed sets that *contain* S. Therefore, by construction we know that $S \subset cl(S)$.

(3) By (1), S is closed and since the closure is constructed as the intersection of *all* closed sets that contain S, we know that one set in the intersection is S itself. Hence $cl(S) \subset S$.

(4) Step (2) and (3) combined tells us that $cl(S) = S$

"⇐" (1) You are given (B), i.e., $cl(S) = S$

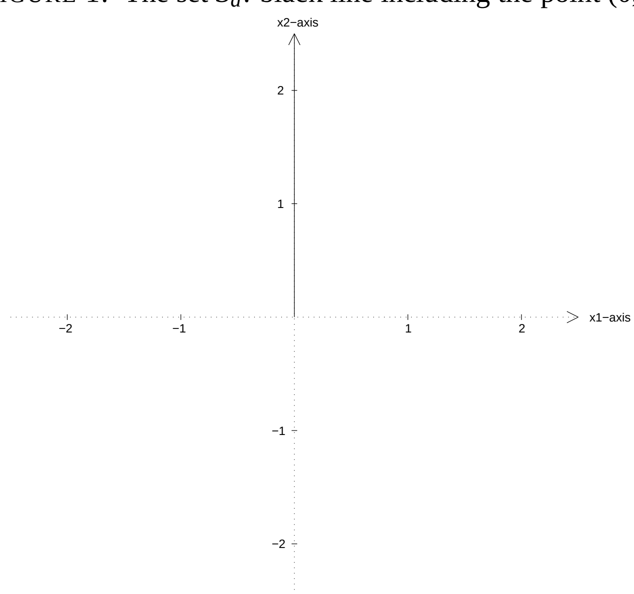
(2) The closure of S is the intersection of all *closed* sets that contain S. From problem 1d we know that the intersection of arbitarily many closed sets is closed again. Hence, $cl(S)$ is a closed set.

(3) By (1) we know that S equals the closure of S and by (2) we know that the closure of S is always closed. Hence, S is closed.

Problem 4

a) The set $S_a = \{(x_1, x_2) : x_1 = 0, x_2 \geq 0\}$ is displayed in figure 1 below. It includes the black line (including the point (0,0)).

FIGURE 1. The set S_a: black line including the point (0,0)



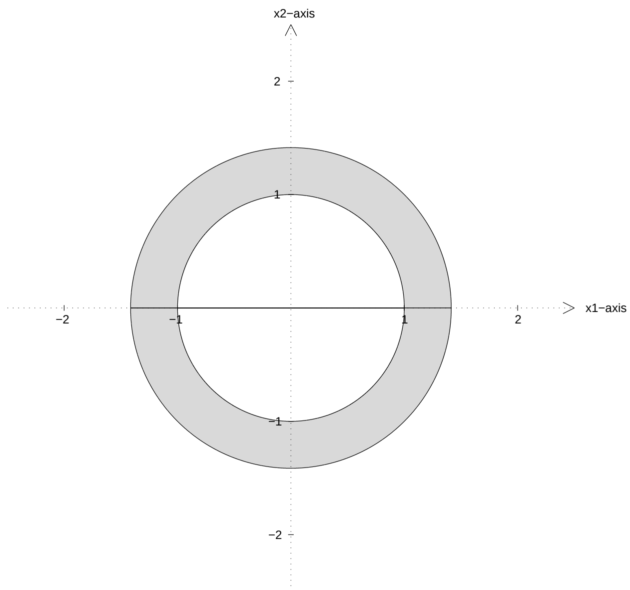
(i) The set S_a is *not* open as any ball around the point $(0,1)$ in the set S_a includes the point $(\frac{\varepsilon}{2}, 1)$ which is not an element of S_a. Hence the ball is not a subset of S_a.

(ii) The set S_a is closed as its complement is open.

(iii) The set S_a is *not* compact as it is not bounded (x_2 can take on any positive value).

b) The set $S_b = \{(x_1, x_2) : 1 \leq x_1^2 + x_2^2 \leq 2\}$ is displayed in figure 2 below. It consists of the shaded area (including the border).

FIGURE 2. The set S_b: shaded area including the border



(i) The set S_b is *not* open as any ball around the point (0,1) in the set S_a includes the point $(0, 1 - \frac{\varepsilon}{2})$ which is not an element of S_b. Hence the ball is not a subset of S_b.

(ii) The set S_b is closed as its complement is open.

(iii) The set S_b is compact as it is bounded and closed.

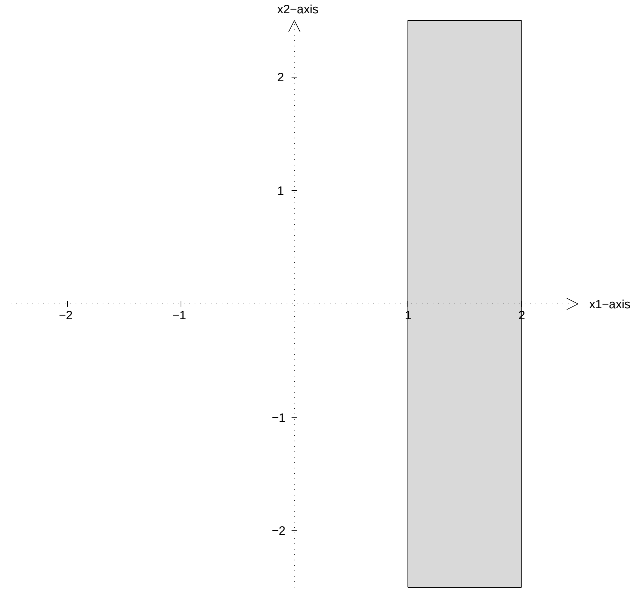
c) The set $S_c = \{(x_1, x_2) : 1 \leq x_1 \leq 2\}$ is displayed in figure 3 below. It consists of the shaded area (including the border).

(i) The set S_c is *not* open as any ball around the point (1,0) in the set S_c includes the point $(1 - \frac{\varepsilon}{2}, 0)$ which is not an element of S_c. Hence the ball is not a subset of S_c.

(ii) The set S_c is closed as its complement is open.

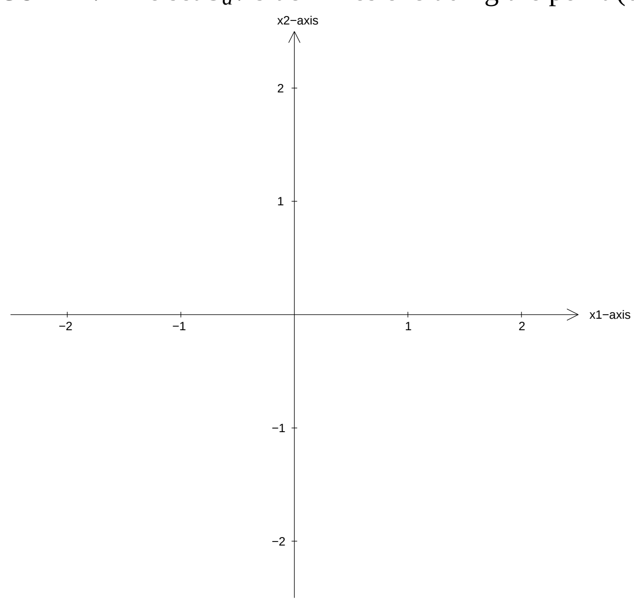
(iii) The set S_c is *not* compact as it is not bounded (x_2 can take on any positive value).

FIGURE 3. The set S_c: shaded area including the border



d) The set $S_d = \{(x_1, x_2) : x_1 = 0 \vee x_2 = 0, \text{ but not both } \}$ is displayed in figure 4 below. It includes the black lines (excluding the point (0,0)).

FIGURE 4. The set S_d: black lines excluding the point (0,0)



(i) The set S_d is *not* open as any ball around the point (1,0) in the set S_d includes the point $(1, \frac{\varepsilon}{2})$ which is not an element of S_d. Hence the ball is not a subset of S_d.

(ii) The set S_d is *not* closed as its complement is *not* open as the point (0,0) is part of the complement and any ball around it includes the point $(0, \frac{\varepsilon}{2})$ which is not an element of the complement. Hence the ball is not a subset of the complement of S_d.

(iii) The set S_d is *not* compact as it is not closed.

Problem 5

a) $int(cl(S)) = int(S)$?

False: Consider the counterexample $X = \mathbb{R}$, $S = \mathbb{R} \setminus \{0\}$ and the Pythagorian metric.

Hence, $cl(S) = \mathbb{R}$ and $int(cl(S)) = int(\mathbb{R}) = \mathbb{R} \neq \mathbb{R} \setminus \{0\} = int(S)$.

b) $cl(S) \cap S = S$?

True: The argument follows

(1) The closure of a set S is the intersection of all closed sets that *contain* S. Therefore, by construction we know that $S \subset cl(S)$.

(2) Using the fact that the inetrsection of a set with a subset always equals the subset we know: $cl(S) \cap S = S$

c) $cl(int(S)) = S$?

False: Consider the counterexample $X = \mathbb{R}$, $S = \mathbb{R} \setminus \{0\}$ and the Pythagorian metric.

Hence, $int(S) = \mathbb{R} \setminus \{0\}$ and $cl(int(S)) = cl(\mathbb{R} \setminus \{0\}) = \mathbb{R} \neq S$.

d) $bd(cl(S)) = cl(bd(S))$?

False: Consider the counterexample $X = \mathbb{R}$, $S = \mathbb{R} \setminus \{0\}$ and the Pythagorian metric.

Hence, $bd(cl(S)) = \{\} \neq \{0\} = cl(bd(S))$.

Problem 6

Prove by induction that $\sum_{k=1}^{n}(2k-1) = n^2$ (*)

(i) *induction initialization*: For n = 1

Left hand side of (*) for n=1: $\sum_{k=1}^{1}(2k-1) = 2*1-1 = 1$

Right hand side of (*) for n=1: $1^2 = 1$

The equality in (*) thus holds for n=1.

(ii) *induction hypothesis*: $\forall n = 1...\bar{n}$ we have $\sum_{k=1}^{n}(2k-1) = n^2$

(iii) *induction step*: $\bar{n} \rightarrow \bar{n}+1$

$$\begin{aligned}
\sum_{k=1}^{\bar{n}+1}(2k-1) &= \sum_{k=1}^{\bar{n}}(2k-1) + \sum_{k=\bar{n}+1}^{\bar{n}+1}(2k-1) \\
&= \underbrace{\sum_{k=1}^{\bar{n}}(2k-1)}_{\bar{n}^2 \texttt{ from (ii)}} + (2(\bar{n}+1)-1) \\
&= \bar{n}^2 + 2\bar{n} + 1 \\
&= (\bar{n}+1)^2
\end{aligned}$$