Handout for Lecture 9 (31/03/2004).

Home Assignment:

Due Wednesday 21/04/2004 in the course mailbox on the 5th floor before 4:45 PM.

6.B.2, 6.B.4, 6.C.1, 6.C.9*, 6.C.16, 6.C.20, 6.E.1*, 7.C.1, 7.E.1

*P. S. Problems with * are not mandatory and will not be graded.*

1 Expected utility

Recall that a lottery is a set of outcomes with corresponding probabilities, $L = (x_1, p_1; ....; x_n, p_n)$. We want to derive a criterion that allows the individuals to choose among lotteries. Assume the individual has preferences over lotteries that satisfy following requirements:

1. (Completeness) For any two lotteries L_1 and L_2, either $L_1 \succeq L_2$ (reads: L_1 is at least as good as L_2 or L_1 is weakly preferred to L_2) or $L_2 \succeq L_1$.

This axiom says that given any two lotteries the individual is always able to choose one. Of course, she might be indifferent, in which case she will be willing to take any. But asked, which is better, she never answers: "I do not know."

2. (Transitivity) For any three lotteries L_1, L_2, and L_3, if $L_1 \succeq L_2$ and $L_2 \succeq L_3$ then $L_1 \succeq L_3$.

This is a very natural consistency requirement. It states that if lottery L_1 is weakly preferred to L_2 and L_2 is weakly preferred to L_3 then L_1 is weakly preferred to L_3.

3. (Independence) Let $L_1 \succeq L_2$ and $\alpha \in [0, 1]$. Then for any lottery $L_3 : \alpha\ L_1 + (1 - \alpha)L_3 \succeq \alpha\ L_2 + (1 - \alpha)L_3$.

To understand the independence axiom suppose you have to lotteries L_1 and L_2 and you weakly prefer L_1 to L_2. Now suppose you are given two choices C_1 and C_2, which are described as follows:

Choice C_1 : Flip an unfair coin with probability of H equal to α and probability of T equal to $1 - \alpha$. If it comes H up, you will face lottery L_1, if it comes up T, you will face lottery L_3.

Choice C_2 : Flip the same coin as in C_1. If it comes H up, you will face lottery L_2, if it comes up T, you will face lottery L_3.

Would you choose C_1 or C_2? Note that if the outcome of the coin flip is T it does not matter which choice had you done, since you face the same lottery L_3 anyway. It only matters if the outcome is H, in which case C_1 will result in you facing L_1 and C_2 in you facing L_2. Since you weakly prefer L_1 to L_2, you should also weakly prefer C_1 over C_2.

Completeness, transitivity, and independence are intuitively appealing requirements. We will call preferences satisfying them CTI preferences. Let $L_1 = (x_1, p_1; ....; x_n, p_n)$ and $L_2 = (x_1, q_1; ....; x_n, q_n)$. (The assumption that both lotteries have the same set of outcomes is without loss of generality. Indeed, let for example, L_1 be $(0, 1/2; 1, 1/2)$ and $L_2 = (2, 1/3; 3, 2/3)$. Those lotteries have different sets of outcomes. Consider, however, $L'_1 = (0, 1/2; 1, 1/2; 2, 0; 3, 0)$ and $L'_2 = (0, 0; 1, 0; 2, 1/3; 3, 2/3)$. Then L'_1 and L'_2 have the same set of outcomes, but L'_1 is essentially the same lottery as L_1, since it differs from it only by probability zero outcomes. Similar, L'_2 is the same lottery as L_2. Therefore, one can always assume that the set of outcomes is the same). It turns out that if the individuals preferences over lotteries satisfy CTI then there exists a function $u(\cdot)$ such that

$$(L_1 \succeq L_2) \Leftrightarrow \sum_{i=1}^{n} p_i u(x_i) \geq \sum_{i=1}^{n} q_i u(x_i). \tag{1}$$

Note that in formula (1) we compare *expected values* of some function of payoffs, that is why (1) is called *the expected utility.* Function $u(\cdot)$ is called Bernoulli utility function. Function $U(\cdot)$ defined by

$$U(L) = \sum_{i=1}^{n} p_i u(x_i) \tag{2}$$

is called von Neumann-Morgenstern utility function.

Note: Bernoulli utility is defined over the monetary payoffs, while von Neumann-Morgenstern utility over the lotteries.

2 Shape of the Bernoulli utility and risk-aversion

The fact that we transform payoffs using $u(\cdot)$ before calculating the expected value allows us to incorporate preferences for risk into our theory. To see how, assume that the Bernoulli utility function is concave and consider a binary lottery $L_1 = (\alpha, x_1; 1 - \alpha, x_2)$ with the expected value $x = \alpha x_1 + (1 - \alpha) x_2$. Let $L_2 = (x, 1)$ be the lottery that gives x with certainty. Now

$$U(L_2) = u(x) = u(\alpha x_1 + (1 - \alpha) x_2). \tag{3}$$

By concavity of the Bernoulli utility function

$$u(\alpha x_1 + (1 - \alpha) x_2) \geq \alpha u(x_1) + (1 - \alpha) u(x_2) = U(L_1). \tag{4}$$

Therefore,

$$U(L_2) \geq U(L_1). \tag{5}$$

Therefore, an individual with concave Bernoulli utility prefers the expected value of the lottery for sure to the lottery itself (the proof can be generalized for more general lotteries). Recall that an individual that prefers the expected value of the lottery for sure to the lottery itself is called risk-averse. Therefore, concavity of the Bernoulli utility is equivalent to the risk-averse behavior. Similar, convexity of the Bernoulli utility is equivalent to the risk-loving behavior, and linearity of the Bernoulli utility is equivalent to the risk-neutral behavior.

3 An Example: buying insurance

Suppose a risk averse consumer has wealth $w > 0$. With probability $q > 0$ she may suffer an accident, in which case her wealth will be reduced to $w - D$, for some $D \in (0, w)$. She has an option to by insurance. If she pays insurance premium x the insurance company will repay her rx in the case of accident for some $r > 1$. Let us find the optimal amount of insurance to buy.

If the consumer purchases amount x of insurance she will have wealth $w - x$ if no accident happens (probability of this is $1 - q$) and wealth $w - x + rx - D$ if the accident happens (probability of this is q). Therefore, her expected utility is

$$U(x) = (1 - q)u(w - x) + qu(w - x + rx - D). \tag{6}$$

Note that $U(\cdot)$ is concave, therefore the F. O. C. are necessary and sufficient for maximum. Therefore, the optimal insurance is (the unique if $U(\cdot)$ is strictly concave) solution to

$$(1 - q)u'(w - x) = q(r - 1)u'(w + (r - 1)x - D). \tag{7}$$

We will call insurance actuarially fair if $r = 1/q$ (that is the firm breaks even on average). Then $q(r - 1) = 1 - q$ and

$$u'(w - x) = u'(w + (r - 1)x - D). \tag{8}$$

Let $u(\cdot)$ be strictly concave. Then, since $u'(\cdot)$ is strictly decreasing

$$w - x = w + (r - 1)x - D \tag{9}$$

$$x = D/r = qD. \tag{10}$$

Note that it leaves the customer with the same wealth $w - qD$ no matter whether the accident happened: a risk-averse individuals insures fully if the price of insurance is actuarially fair.

4 Stochastic dominance

If we know an individual's Bernoulli utility then we can compare any two lotteries from her point of view. Now I am going to ask: Given two monetary lotteries under what conditions will I be able to say that any (risk-averse) individual prefers one to second, provided she prefers more to less.

Definition *Lottery* $L_1 = (x_1, p_1; ....; x_n, p_n)$ *is said to* <u>*first order stochastically dominate*</u> *(FOSD) lottery* $L_2 = (x_1, q_1; ....; x_n, q_n)$ *if for any increasing Bernoulli utility function* $u(\cdot)$

$$\sum_{i=1}^{n} p_i u(x_i) \geq \sum_{i=1}^{n} q_i u(x_i). \tag{11}$$

Our next objective is to derive a criterion, which will allow us to decide whether one lottery FOSD the other.

Definition *Function* $F(\cdot)$ *defined by*

$$F(z) = \Pr(x < z) \tag{12}$$

is called a cumulative distribution function for random variable x.

Let $L = x = (x_1, p_1; ....; x_n, p_n)$ and assume without loss of generality that $x_1 < x_2 < .... < x_n$. Then

$$F(z) = 0, \text{ for } z < x_1, \tag{13}$$

$$F_L(z) = \sum_{i=1}^{k} p_i, \text{ for } x_k < z \leq x_{k+1}, \tag{14}$$

$$F(z) = 1, \text{ for } z \geq x_n. \tag{15}$$

L_1 FOSD L_2 iff $F_{L_1}(z) \leq F_{L_2}(z)$, that is the probability that the outcome is lower then any fixed level is smaller for lottery L_1.

Example $L_1 = (0, 1/6; 1, 1/3; 2, 1/2)$ and $L_2 = (0, 1/3; 1, 1/3; 2, 1/3)$Then L_1 FOSD L_2. Let first establish it using the definition. We have to check that

$$\frac{1}{6}u(0) + \frac{1}{3}u(1) + \frac{1}{2}u(2) \geq \frac{1}{3}u(0) + \frac{1}{3}u(1) + \frac{1}{3}u(2). \tag{16}$$

for any increasing $u(\cdot)$. But inequality FOSD is equivalent to

$$u(2) \geq u(0). \tag{17}$$

Definition *Let lotteries* L_1 *and* L_2 *have the same expected value (mean). Lottery* $L_1 = (x_1, p_1; ....; x_n, p_n)$ *is said to* <u>*second order stochastically dominate*</u> *(SOSD) lottery* $L_2 = (x_1, q_1; ....; x_n, q_n)$ *if for any increasing concave Bernoulli utility function* $u(\cdot)$

$$\sum_{i=1}^{n} p_i u(x_i) \geq \sum_{i=1}^{n} q_i u(x_i). \tag{18}$$

Our next objective is to derive a criterion, which will allow us to decide whether one lottery SOSD the other.

Definition *A lottery L_2 is said to be obtained from lottery L_1 by a mean-preserving increase of risk if it is obtained by replacing an outcome x_i in lottery L_1 by a lottery with mean x_i.*

Example Let $L_1 = (0, 1/3; 1, 1/3; 2, 1/3)$ and $L_2 = (0, 1/2; 2, 1/2)$. Lottery L_2 is obtained from L_1 by replacing the outcome 1 with a lottery $(0, 1/2; 2, 1/2)$. For example, L_1 may be realized as: through a dice, if $\{1, 2\}$ get nothing, if $\{3, 4\}$ get one, if $\{5, 6\}$ get $\dot{2}$. L_2 provides the same payoff if the outcomes are $\{1, 2, 5, 6\}$ if the outcome is 3 or 4, through a coin and get nothing if H and two if T.

Distribution L_1 SOSD L_2 iff L_2 is obtained from lottery L_1 by a mean-preserving increase of risk. Intuitively L_2 contains more risk, since a certain outcome was replaced by the lottery. Therefore, any risk-averse individual will prefer L_1 to L_2.