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Dynamical formation of a Bose–Einstein condensate

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Abstract

We explain how a condensate forms in finite time by a selfsimilar blow-up of the solution of the relevant quantum Boltzmann kinetic equation for a dilute quantum Bose gas. The condensate, once it is there, keeps exchanging mass with the rest of the distribution until equilibrium is reached, as described by a version of the kinetic equation that includes the existence of this condensate. © 2001 Published by Elsevier Science B.V.

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1. Introduction and quantum Boltzmann equation

Soon after the final conception of non-relativistic quantum theory, Nordheim [1] proposed a Boltzmann like quantum kinetic theory for bosons and fermions, describing in particular relaxation to equilibrium. This kinetic equation describes the dynamics of the momentum distribution that is also the Wigner transform of the one-particle density matrix. Below, we address the question of the formation of a singular equilibrium distribution as a solution of the Boltzmann–Nordheim (BN for short later on) equation after a finite time. We explain how, if the initial number density exceeds a critical threshold, some solutions of the kinetic equation may blow-up at a finite time t_* (depending on the initial conditions). This time t_* is the incipient time for the BE condensate (BE stands for Bose–Einstein). In the case of the BN equation for bosons, it seems obvious that the piling up of particles near zero momentum is a manifestation of the BE condensation. However, the connection is not that obvious, since the collapse at zero momentum is a dynamical process, without any direct link with the physics behind the BE condensation.

Once the condensate is formed, its mass can still evolve by exchange with the thermal background, until the global equilibrium BE distribution is reached for the given conditions of mass and energy. The growth of a singular part in the momentum distribution is an indication that a condensate is formed, in some sense. However, this cannot mean that phase correlations with an infinite range set in after a finite time. This would imply the unphysical assumption that for an infinite system the "information" (of phase) propagates at infinite speed. We discuss in Section 4 this question of phase coherence at large distances, and the way it appears dynamically.

A collapse of the distribution density has been studied before in the context of a nonlinear Fokker–Planck or Kompaneets equation [2]. Although, it shares some features of the present problem, there are some important differences. In particular, the Kompaneets equation does not preserve energy, because it describes the evolution to equilibrium of a Bose gas with a

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constant temperature background. However, it shows nicely transfer of mass through the energy spectrum as we shall present here. Similar ideas were developed in the context of BN equation by Levich and Yakhot [3,11] and Kagan and collaborators [4] but with conclusions different from ours.

The question of the incipient phase singularity has been investigated numerically by Semikoz and Tkachev [5] and our results agree with this work. Below, we address a new question, however, the way the mass of the condensate grows after the collapse time (here "condensate" just means the singular piece of the momentum distribution, related in a rather subtle way to the large scale coherence of the condensate in the true sense, see Section 4). This law of growth is probably the most relevant information, as it can likely be related to physical observations. It rests upon a detailed understanding of the analytical structure of the finite time singularity that is based upon the observation that the exponent is a nonlinear eigenvalue of the similarity equation for collapse. The application of the BN equation to this problem meets the following difficulty (a second one shall be discussed in Section 3): as the formation of the condensate is predicted to occur through a solution with a finite time singularity, the rate of evolution of this solution diverges like the inverse of the time remaining until the singularity, which makes the kinetic theory invalid when this time scale becomes shorter than the period associated with free particle motion by the Planck–Einstein correspondence. Because of the low density assumption, this breaking of the validity of the kinetic theory occurs at a late stage of the blow-up process if $fn^{1/3} \ll 1$ as we show at the end of Section 2.

The BN kinetic equation for a homogeneous distribution in space (we shall discuss briefly non-homogeneous condensation at the end) reads for bosons

$$\begin{aligned}\partial_t w_{p_1}(t) &= \mathrm{Coll}[w] \\ &\equiv \int \mathrm{d}^3 p_2\, \mathrm{d}^3 p_3\, \mathrm{d}^3 p_4 W_{p_1,p_2;p_3,p_4} \\ &\quad \times (w_{p_3} w_{p_4}(1+w_{p_1})(1+w_{p_2}) \\ &\quad - w_{p_1} w_{p_2}(1+w_{p_3})(1+w_{p_4})), \end{aligned} \tag{1}$$

where $w_p(t)$ can be seen as the probability distribution for the momentum, [1] m is the atomic mass, $2\pi\hbar$ the Planck's constant. Moreover,

$$\begin{aligned} W_{p_1,p_2;p_3,p_4} &= \frac{1}{m\hbar^3}(|f_{p_1-p_2}|^2 + |f_{p_2-p_1}|^2) \\ &\quad \times \delta^{(3)}(p_1+p_2-p_3-p_4) \\ &\quad \times \delta^{(1)}(p_1^2+p_2^2-p_3^2-p_4^2) \end{aligned}$$

gives back the Boltzmann original writing once the integrals over p_3 and p_4 are carried out, f being the scattering length taken as constant for the low momentum s-wave scattering. The Wigner distribution is normalized by $(1/\hbar^3)\int \mathrm{d}^3 p w_p(t) = n \equiv N/V$, N is the total number of particles and V the volume of the enclosure. We shall take $\hbar = m = 1$ throughout the analysis.

An H-theorem shows that solutions of (1) relax to

$$w_p^{\mathrm{eq}} = \frac{1}{\mathrm{e}^{(p^2/2-\mu)/T} - 1}$$

(T is the absolute temperature in energy units) constrained by the conservation of the number of particles and of the energy. Take the initial condition $w_p(t=0) = A\,\mathrm{e}^{-p^2/\gamma}$. The relaxation to equilibrium preserves $\int_0^\infty p^\alpha w_p p^2\, \mathrm{d}p$ with $\alpha = 0$ and 2 which yields a relation between A and the dimensionless chemical potential μ/T:

$$A = (\zeta_{3/2}(\mathrm{e}^{\mu/T}))^{5/2}(\zeta_{5/2}(\mathrm{e}^{\mu/T}))^{-3/2} \tag{2}$$

with $\zeta_s(z) = \sum_{n=1}^{\infty}(z^n/n^s)$, incomplete Riemann ζ-function.

At low densities (small A) μ is negative as in an ideal classical gas. As A increases μ increases too, until a critical value: $A_c = \zeta_{3/2}(1)^{5/2}/\zeta_{5/2}(1)^{3/2} = 7.0992\ldots$, where μ vanishes. However, if $A > A_c$, it is not possible to satisfy (2) with μ negative, and the transition predicted by Einstein in 1924 [6] occurs. We have computed the relation between the chemical potential obtained from the numerical solution of (1) at very late times and the initial amplitude A in order to

[1] Note that the Wigner functions are real but not necessarily positive, however, if $w_p(t=0) > 0$, then the BN equation keeps it positive at any further time (at least for bosons). For fermions, this probability would have to stay between 0 and 1 to keep clear of mathematical troubles.

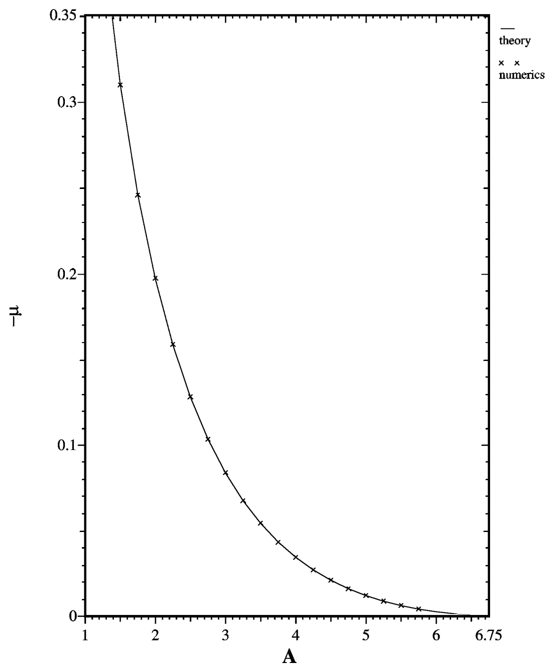

Fig. 1. The equilibrium chemical potential μ as a function of the initial amplitude A. The crosses represent the numerical values obtained from the evolution of the BN equation, while the continuous line traces the theoretical curve (2).

test our numerical code for solving (1), which worked very well as we see in Fig. 1.

The question now is: let $w_p(t=0)$ (e.g. the above form) be a smooth initial (non-equilibrium) condition for (1), what is the further evolution of $w_p(t)$? In particular, whenever A is larger than the critical amplitude A_c. We shall explain first how to describe the finite time singularity by means of a selfsimilar solution of the full kinetic equation. [2]

2. Dynamics before collapse

If $A > A_c$, we expect condensation to zero momentum, that is the spontaneous occurrence of a singularity in the solutions of (1) for $p = 0$, a singularity leading to a solution of the type $w_p = n_0\delta^{(3)}(p) + \varphi_p$, φ_p is a smooth function, an interesting phenomena on its own. Therefore we expect that just before the singularity the occupation number of small momenta becomes very large, $w_p \gg 1$, which allows to neglect, for that purpose, the quadratic term in Eq. (1) with respect to the cubic one, a remark that has been already made in this context [3,4,11], but with different conclusions from ours as we said. This yields a simpler "degenerate" form of the kinetic equation. (This kinetic equation has been thoroughly studied in the context of nonlinear wave interaction and "weak-turbulence". For a review and references, see [7]. However, the time dependent selfsimilar solution exposed in the present note does not seem to have been considered before.) [3] Let $\epsilon = \frac{1}{2}p^2$, and $\tilde{W}_{\epsilon_1,\epsilon_2;\epsilon_3,\epsilon_4} = (f^2/m\hbar^3)\min\{\sqrt{\epsilon_1}, \sqrt{\epsilon_2}, \sqrt{\epsilon_3}, \sqrt{\epsilon_4}\}$:

$$\begin{aligned}\partial_t w_{\epsilon_1}(t) &= \mathrm{Coll}_3[w] \\ &\equiv \frac{1}{\sqrt{\epsilon_1}}\int_D \mathrm{d}\epsilon_3\,\mathrm{d}\epsilon_4\,\tilde{W}_{\epsilon_1,\epsilon_2;\epsilon_3,\epsilon_4}(w_{\epsilon_3}w_{\epsilon_4}w_{\epsilon_1} \\ &\quad + w_{\epsilon_3}w_{\epsilon_4}w_{\epsilon_2} - w_{\epsilon_1}w_{\epsilon_2}w_{\epsilon_3} \\ &\quad - w_{\epsilon_1}w_{\epsilon_2}w_{\epsilon_4}).\end{aligned} \tag{3}$$

Since $\epsilon_2 = \epsilon_3 + \epsilon_4 - \epsilon_1$ must be positive, one integrates in a domain D such as $\epsilon_3 + \epsilon_4 > \epsilon_1$. The equilibrium solution of this equation follows from the maximization of entropy and is $w_\epsilon = T/(\epsilon - \mu)$. This is a formal solution only, because it does not yield a converging expression for the energy nor even for the total mass. [4] For finite total mass and energy this solution should be a function spreading forever in momentum space [8], a spreading stopped in the full BN equation by the quadratic terms in Coll$[w]$. Zakharov has found two other stationary solutions

$$w_\epsilon = Q^{1/3}\epsilon^{-3/2}, \qquad w_\epsilon = J^{1/3}\epsilon^{-7/6}. \tag{4}$$

[2] We investigated numerically this question and found results in complete agreement with the selfsimilar solution described below. We plan to report the details of these numerical investigations (methods and results) in a future extended publication.

[3] As shown by Zakharov [12], and by Carleman before [13] for an isotropic momentum dependence of the distribution function one may integrate both sides of (1) in solid angles, this allows one to write the simpler form (3) for w_ϵ, which is better for the numerics.

[4] Generally speaking, this kind of divergence at "large momentum/energy" is irrelevant for the present analysis, because for this momentum, the cubic approximation to the collision operator is not valid anymore, so that the power solution for w_ϵ merge with solutions "at large" (actually non-small) energies that take care of the convergence of the integrals for mass and energy.

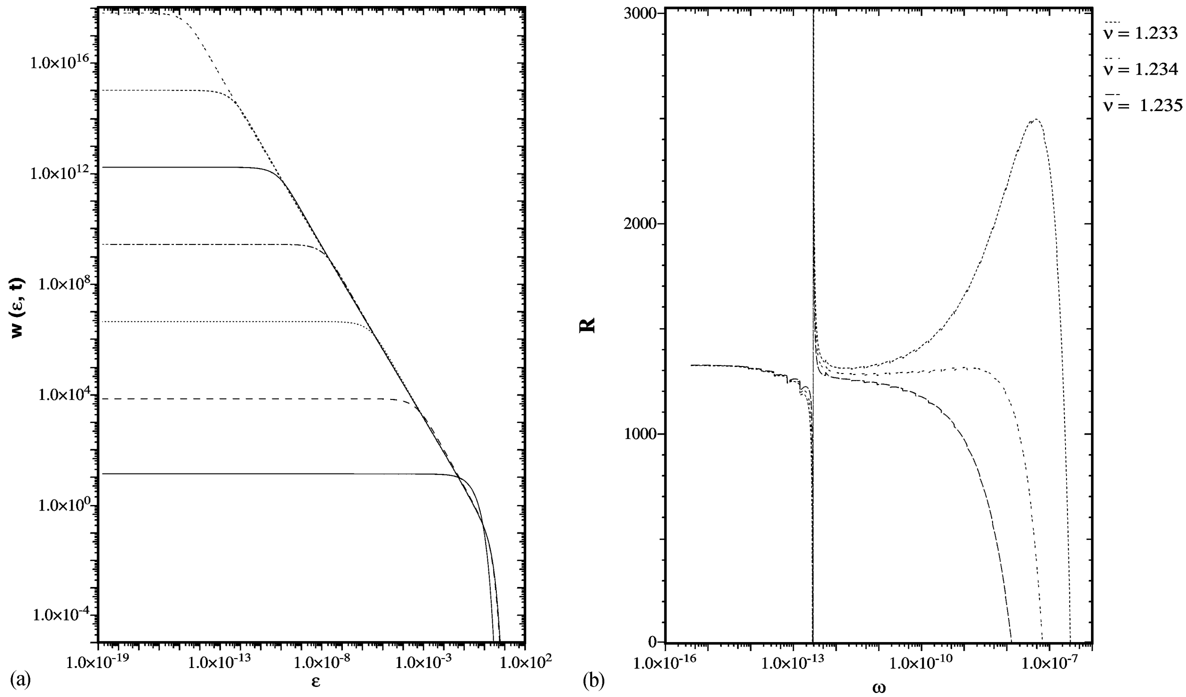

Fig. 2. (a) The distribution function $w(\epsilon, t)$ (at times chosen for successive increase of $w(0, t)$ by a factor 5). The different time plots show a clear selfsimilar evolution. One sees the build-up of the power law distribution $\epsilon^{-1.234}$ from the large energies to the small ones. (b) The ratio between the right- and left-hand side of Eq. (6). One sees that the best agreement is for $\nu = 1.234$. The existence of a plateau proves that one has the right "nonlinear eigenmode" and the right "nonlinear eigenvalue" ν.

Here $Q(/J)$ is the energy(/mass) flux in momentum space per unit time. Those solutions are derived by a Kolmogorov-like analysis, for Q and J constant. It does not seem possible, however, to use this kind of Kolmogorov-like solution for the present problem, because we expect the collapse to be a dynamical process, so that stationary solutions can help at best to understand qualitatively the transfer of mass and energy through the spectrum. In particular, as shown later on, the actual exponents for the selfsimilar solution do not follow from simple scaling estimate.

We remark that because of its structure (in particular because the right-hand side of (3) is cubic homogeneous in w_ϵ), Eq. (3) admits a selfsimilar dynamical solution which accumulates particles at zero momentum. The selfsimilar solution has the form ($\tau = t_* - t$):

$$w_\epsilon(t) = \beta^{-1/2}\tau^{-\alpha}\phi(\epsilon\tau^{-\beta}) \tag{5}$$

as $t \to t_*$, $\alpha, \beta > 0$. Putting (5) into (3) and imposing that the left- and right-hand sides are of the same order as $\tau \to 0$, one gets that $\beta = \alpha - \frac{1}{2}$, the integro-differential equation for ϕ becomes

$$-(\nu + \omega\partial_\omega)\phi(\omega) = \mathrm{Coll}_3[\phi(\omega)], \tag{6}$$

where $\omega = \epsilon\tau^{-\beta}$ and $\nu = \alpha/\beta$ is the only remaining free parameter. As shown below, this parameter is a nonlinear eigenvalue of (6) allowing to satisfy the boundary conditions $\phi(0)$ finite yet to be determined and $\phi(\omega) \approx \omega^{-\nu}$ as $\omega \to \infty$ with a convenient choice of normalization for ϕ.

We observed in our numerics that such a power law spectrum $\phi(\omega) \sim 1/\omega^\nu$ was established (for large numerical values of ω) without mean flux of mass or energy on the energy scale. The "observed" (see Fig. 2) value is roughly $\nu \approx 1.234(1)$ which differs significantly from $\frac{7}{6}$ and $\frac{3}{2}$ that would follow from the scaling properties of the solutions at constant mass or energy flux.

Besides a direct numerical attack (see Fig. 2a), it seems difficult to get much analytical information concerning solution(s) of (6). We shall nevertheless present some remarks relevant to this problem. One may construct order by order a Laurent expansion of $\phi(\omega)$ for large ω, beginning as $1/\omega^{\nu}$ and then putting this first term into Coll_3. The beginning of this expansion reads

$$\phi(\omega) = \frac{1}{\omega^{\nu}} - \frac{C(\nu)}{2(\nu-1)\omega^{3\nu-2}} + \mathrm{O}\left(\frac{1}{\omega^{5\nu-4}}\right) \qquad (7)$$

with $C(\nu)$ defined by the action of the collision operator Coll_3 on a power law distribution $\mathrm{Coll}_3[\omega^{-\nu}] \equiv C(\nu)\omega^{-3\nu+2}$. The function $C(\nu)$ is positive for $\nu \in [1, \frac{7}{6}]$, negative for $\nu \in [\frac{7}{6}, \frac{3}{2}]$ and positive again for $\nu > \frac{3}{2}$. One sees now why it is not possible to get $\nu = \frac{7}{6}$ nor $\frac{3}{2}$ as it should follow from (4), because the next order and any higher order correction vanishes since $C(\nu)$ is zero for both cases, and the Laurent expansion at large ω stops there.

Therefore, this kind of solution (7) is already singular at $\omega = 0$, although we want to study evolution of a solution remaining finite at $\epsilon = 0$ at any time less than t_*, which implies $\phi(\omega = 0)$ finite. One may expect to push the Laurent expansion in order to capture better and better the behavior near $\omega = 0$. As we said, the resulting series will diverge almost always when approaching $\omega = 0$ which is a singular point, because near $\omega = 0$ it is possible to expand the solution of (6) in the form $\phi = a(\nu)\omega^{-7/6} + \cdots$, the entire function $a(\nu)$ being completely determined by the outer matching (this defines the asymptotic behavior of the solution). The condition $a(\nu) = 0$ fixes ν.

Supposing that the integral equation (6) has a smooth solution that satisfies all the right boundary conditions, it describes a collapsing solution of the original kinetic equation. The distribution function at the peak scales like $w(\epsilon = 0) \sim \tau^{-\alpha}$; the energy-stretching of the peak: $\epsilon_0 \sim \tau^{\beta}$; the flux of particles: $j_0 \sim \tau^{-\gamma}$; the flux of energy: $Q \sim \tau^{\delta}$; and the density of particle at the peak (that is with an energy less than ϵ_0): $n_0 \sim \tau^{\xi}$. All these exponents can be deduced from ν by simple algebraic manipulations.

In the following table, we compare the theoretical values from the formula (second row) together with $\nu = 1.234$, and the direct numerical values:

Exponent	Relation with ν	For $\nu = 1.234$	Numerics
α	$\nu/2(\nu-1)$	2.637	2.639
β	$1/2(\nu-1)$	2.137	2.139
γ	$3(\nu-\frac{7}{6})/2(\nu-1)$	0.4316	0.4317
δ	$3(\frac{3}{2}-\nu)/2(\nu-1)$	1.705	1.707
ξ	$(\frac{3}{2}-\nu)/2(\nu-1)$	0.568	0.571

The numerical solution of (3) is in excellent agreement with this scenario, in particular with the exponents for the scaling laws for the collapse concerning their relation to ν. On the other hand, one can check that the numerical selfsimilar distribution, once written as in (5) yields a function ϕ that satisfies numerically Eq. (6) (see Fig. 2b).

The collapse time t_* depends on the initial conditions. Therefore one expects a dependence of t_* on the threshold to A_c when two and three body collisions are included (in the case of only three body collisions term as in (3) one has always a singularity in a time $t_* \sim A^{-2}$). Numerically, we have found $t_* \sim |A - A_c|^{-\eta}$ with $\eta = 0.4$. This time t_* is about the time when quadratic and cubic terms become of the same order in the full BN equation.

Notice that the flux of particles towards the origin diverges as t goes to t_*, whereas the number of particles in the condensate remains zero.[5] This means that the true formation of a condensate starts just after t_*. We shall explain next what is predicted by the kinetic equation after the condensate is formed. Finally, we note that the exponent ν must be larger than $\frac{7}{6}$ because one needs to have an infinite flux of particles through the peak in order to ensure the finite time singularity. In the usual (non-quantum) Boltzmann equation this is not possible: the power law solution with a constant flux of matter is $w_{\epsilon} = (J/f^2)^{1/2}\epsilon^{-7/4}$ and possess an infinite mass at the peak at $\epsilon = 0$. Therefore the usual Boltzmann equation for hard spheres does not present

[5] This flux of particles is practically across an energy surface that shrinks like τ^{β}, therefore there is no contradiction: the flux diverges, but it accumulates mostly outside of the origin, since it is across a non-constant barrier on the energy scale.

a finite time singularity. However, when the two particle interaction decreases at large distances like $\sim 1/r^l$ one has an explicit dependence of the scattering amplitude on the relative velocity of collisions. There are reasons to believe that for $l < 4$, the usual Boltzmann equation could present a finite time collapse as the one described here, because the collision frequency diverges at small speed.

Ending this section let us discuss the breaking of the validity of the kinetic theory near the blow-up. Let τ be the time left until blow-up ($\tau_{\text{blow-up}} = 0$), n be the total number density, and f the scattering length. The mean free flight time for the core of the energy spectrum is $t_{\text{mfp}} = (1/nf^2)(m/\epsilon_{\text{Th}})^{1/2}$, where $\epsilon_{\text{Th}} \sim p^2/2m$ is the average kinetic energy per particle (a constant). Assuming this energy to be of the order of magnitude of the one at the BE transition, one has $t_{\text{mfp}} = m/\hbar n^{4/3} f^2$. The BN kinetic theory applies if the typical evolution time until blow-up, τ, is still much larger than $\hbar/\epsilon_0(\tau)$, where $\epsilon_0(\tau)$ is the average energy of particles taking part in this blow-up ($\epsilon_0(\tau) \to 0$ as $\tau \to 0$). We have shown in this section that $\epsilon_0(\tau) \sim \epsilon_{\text{Th}}(\tau/t_{\text{mfp}})^\beta$ ($\beta > 0$). Therefore, the BN kinetic theory applies for $\tau > \tau_{\text{cr}}$ with $\tau_{\text{cr}} = \hbar/\epsilon_0(\tau_{\text{cr}})$. From the estimates given above, the inequality $\tau_{\text{cr}} \ll t_{\text{mfp}}$ is equivalent to $fn^{1/3} \ll 1$, precisely the condition for a dilute gas. Therefore, in this dilute gas limit, the BN kinetic theory remains physically sound in the time interval $[t_{\text{mfp}}, \tau_{\text{cr}}]$ before blow-up.

3. Dynamics after collapse

At the singularity time, if our scenario of selfsimilar collapse holds, as seems to be confirmed by our numerical studies, the system is not yet at equilibrium, and some exchange of mass between the condensate and the rest is necessary to reach full equilibrium, because the mass inside the singularity is still zero at $t = t_*$. It happens that this exchange of mass can be described by extending the full kinetic equation to singular distributions, something that does not seem to have been noticed before to the best of our knowledge. As $w(\epsilon = 0)$ and the flux of matter diverge at $t = t_*$, let us consider the following ansatz for times larger than t_*: the distribution function behaves as

$$w_p(t) = n_0(t)\delta^{(3)}(p) + \varphi_p,$$

φ_p smooth function, and $n_0(t_*) = 0$. Now, the collision integral in (3) splits as $\text{Coll}[w_p] = j_0(t)\delta^{(3)}(p) + \tilde{\text{Coll}}[\varphi_p]$, where $j_0[\varphi] = \int_{S(0^+)} \sqrt{\epsilon_1}\, \text{d}\epsilon_1\, \text{Coll}[w_{\epsilon_1}] = -\int_0^\infty \sqrt{\epsilon_1}\, \text{d}\epsilon_1\, \tilde{\text{Coll}}[\varphi_{\epsilon_1}]$, and $\tilde{\text{Coll}}[\varphi]$ is for the exact BN collision term of (1). This special form says that there is a macroscopic flux of particles towards the zero momentum, which is directly related to a $j_0^{1/3}\epsilon^{-7/6}$ term as the dominant behavior of φ_ϵ near $\epsilon = 0$. This exchange term (between particles in the condensate and excited states) has not been considered previously. Putting this ansatz into (1) one gets, after splitting the terms with non-zero integral in a small sphere around $p_1 = 0$:

$$\partial_t n_0(t) = j_0 + n_0(t)\, \text{Coll}_2[\varphi], \quad \text{where}$$
$$\text{Coll}_2[\varphi] = \int_{2,3,4} \delta_{2;3,4}(\varphi_{p_3}\varphi_{p_4} - \varphi_{p_2}(\varphi_{p_3} + \varphi_{p_4} + 1)), \tag{8}$$

$$\partial_t \varphi_{p_1}(t) = \tilde{\text{Coll}}[\varphi] + n_0(t)\, \tilde{\text{Coll}}_2[\varphi], \quad \text{where}$$
$$\begin{aligned}\tilde{\text{Coll}}_2[\varphi] = {} & \int_{3,4} \delta_{1;3,4}(\varphi_{p_3}\varphi_{p_4} - \varphi_{p_1}(\varphi_{p_3} + \varphi_{p_4} + 1)) \\ & + 2\int_{2,4} \delta_{1,2;4}(\varphi_{p_4}(\varphi_{p_1} + \varphi_{p_2} + 1) \\ & - \varphi_{p_1}\varphi_{p_2}).\end{aligned} \tag{9}$$

We used the notations $\int_{2,3} = \int \text{d}^3 p_2\, \text{d}^3 p_3$, and $\delta_{2;3,4} = W_{0,p_2;p_3,p_4}\delta^{(3)}(p_2 - p_3 - p_4)\delta^{(1)}(p_2^2 - p_3^2 - p_4^2)$, and so forth. These coupled equations conserve mass and energy and a H-theorem applies. [6] For very short times, that is $|t - t_*| \ll t_{\text{B}}$ (see below for the definition of t_{B}), it is possible to calculate a selfsimilar solution of the form: $n_0 = K(-\tau)^\sigma$, together with the same kind of expression for φ as before. The exponent $\sigma = (\frac{3}{2} - \nu)/2(\nu - 1)$, while α and β remain

[6] The coupled set (8) and (9) relax as $t \to \infty$, to the equilibrium solution found by Einstein [6]:

$$\varphi_p^{\text{eq}} = \frac{1}{\text{e}^{p^2/2T} - 1},$$

and n_0 fixed by mass conservation. Notice, however, that interactions modify this picture (see below).

the same as before. For times just after t_*, one expects that the function φ_ϵ will be very close to the function before the collapse "far" from zero energy, since it changes infinitely fast near the origin only. Therefore, by continuity this imposes that φ and ϕ behave in the same way for large ω, and this implies that the coefficient ν is the same as before. The constant K and $a(\nu)$ (which is no longer zero) are fixed by a complicated set of integro-differential equations following directly from the most singular terms in (8) and (9).

The short time selfsimilar behavior merges at later time with a relaxation behavior tending to equilibrium. The full system (8) and (9) describes the relaxation towards a constant value of the density of condensate n_0, while the flux j_0 and different collisional terms vanish leading to an equilibrium distribution for φ_ϵ, which is the Bose factor with zero chemical potential.

Ending this section, we notice that the BN equation is not uniformly valid in its original form after the formation of a condensate, since the appearance of such a structure changes the energy spectrum at low momenta, as shown by Bogoliubov [9], although the kinetic equation assumes that, besides collisions, the particles have a purely ballistic spectrum. For a dilute gas (where the quantum kinetic equation is valid) the BN equation applies for most particles, since the Bogoliubov renormalization of the energy concerns a narrow energy domain. This would change the late stages of the condensation only and would apply anyway to the range of energies $\epsilon \ll \epsilon(\tau_{\mathrm{cr}})$, where τ_{cr} has been defined before. It turns out that the typical Bogoliubov time scale: $t_{\mathrm{B}} = m/\hbar f n_0$ appears at a later stage because it is much longer than the mean free path t_{mfp} if $n^{4/3} f \gg n_0$, which is possible for a dilute gas $f n^{1/3} \ll 1$ as long as $n_0 \ll n$.

4. Collapse and build-up of long range correlations

As explained before, the collapse, as it results from the singular solution of the BN kinetic equation, cannot give the full physical picture. This is because the quantum kinetic theory, as representing the true many body dynamics, relies upon the assumption that the time scale for the evolution of w_p is much longer than the time scale set by the Planck period of the motion of a free particle. In the collapse phenomenon, this assumption becomes clearly untrue at some stage, since the typical time scale for the evolution of w_p is of order τ, the time until the collapse, and so becomes as short as wanted, although the Planck period associated to particles of low momentum becomes infinitely large when this momentum tends to zero. This puts a constraint on the time when the kinetic theory looses meaning close to the collapse time. Therefore, the standard kinetic approach fails at time scales of order or shorter than τ_{cr} with $\tau_{\mathrm{cr}} \sim \hbar/\epsilon_0(\tau_{\mathrm{cr}})$ that is to be combined with the estimate $\epsilon_0(\tau) \sim \epsilon_{\mathrm{Th}}(\tau/t_{\mathrm{mfp}})^\beta$, β is positive, to yield

$$\tau_{\mathrm{cr}} \sim t_{\mathrm{mfp}}(f^2 n^{2/3})^{1/(\beta+1)}.$$

For times far closer to the collapse time than τ_{cr}, it becomes inconsistent to use the kinetic theory in its standard form to describe what happens in the selfsimilar core of the distribution. In this range of times, as well as later on, the growth of the coherence length can be described by using arguments borrowed from Pomeau [10]. The starting point is to assume that there is a certain mass density in the condensate, approximately uniform in space, but with a random phase without long range order. Moreover, this can be represented by the dynamics of a classical field, since the occupation number at small momenta is very large. This phase relaxes, at least after some transient according to the equations of fluid mechanics derived for long wave perturbations, namely the Bernoulli equations for a compressible fluid. The energy is transferred to small scales now by a steepening of the gradient of the phase at random places, which yields finally a typical time scale. Accordingly, the phase of the "condensate" becomes uniform on space scales $L(t)$ growing with time like $L(t) \sim (\hbar t/m)^{1/2}$.

Consider now what happens at times much later than the blow-up time. Because the law of growth of the coherence length $L(t)$ is independent upon the density, one expects that this length represents at time t the coherence length scale for the phase, $t = 0$ being the time of the singularity, after which the long range correlations begin to build up. Near the blow up time the situation is probably more complicated, although one can devise a simple estimate for the phase coherence length that should be of order of the de Broglie wavelength associated to the Planck period τ_{cr}.

5. Comments and conclusions

We propose a scenario for the formation of a condensate of BE particles obeying the BN kinetic equation. This scenario consists in a blow-up of the distribution function at zero momentum. However, the total mass density of this singularity is zero at the time of the singularity, so that one needs to feed this "condensate" in order to reach equilibrium at later times. This is a very important point, because we show that the kinetic equation predicts an exchange of mass between the condensate and the rest of the system when the gas is out of equilibrium. Finally, if there is spatial dependence one needs to add a $((p_1/m)\cdot\nabla_r w_{p_1} - (\nabla U/m)\cdot\nabla_{p_1} w_{p_1})$ term to the left-hand side of (1). This leads to a spatial localization of the finite time singularity of the type $r \sim \tau^{1+\beta/2}$, $p_1/m\cdot\nabla_r w_{p_1}$ being the most singular term if the potential energy grows faster than $U(r)\sim r^{4/(2+\beta)} \approx r^{0.97}$. Another remark is of interest: the connection between the finite singularity at zero momentum of the solution of the BN equation is not so directly related to the existence of a condensate in the equilibrium theory. Actually, it is more like a property of the dynamical equation itself. In particular, the power law for the distribution close to zero momenta is not the inverse ω typical of the Bose factor. Therefore one expects that at some later time, after the collapse this power law will switch from the $\omega^{-7/6}$ typical of the kinetic regime to the ω^{-1} typical of the equilibrium regime.

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