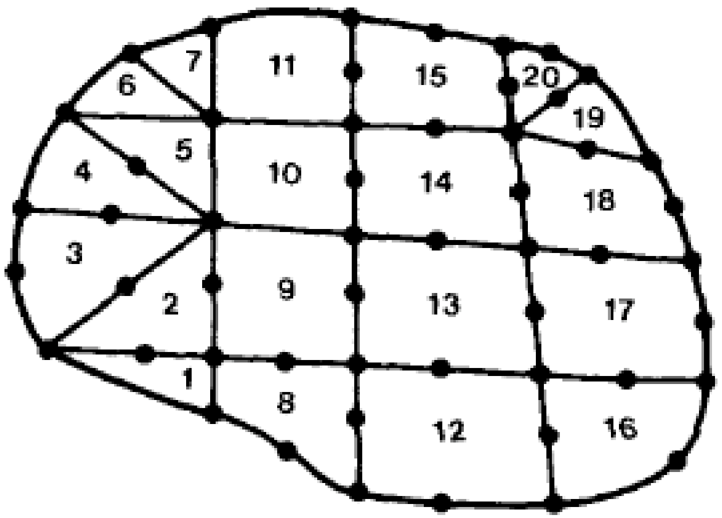
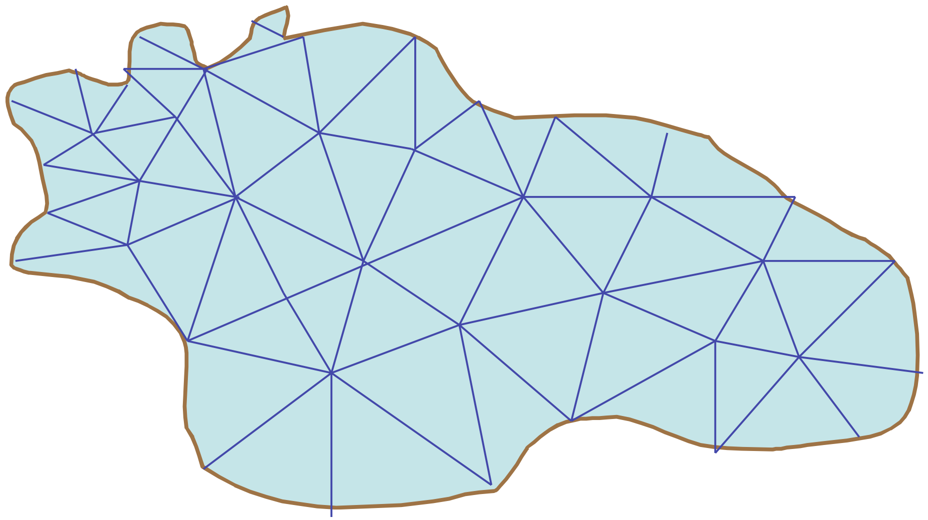
ELEMENTOS FINITOS
2-D

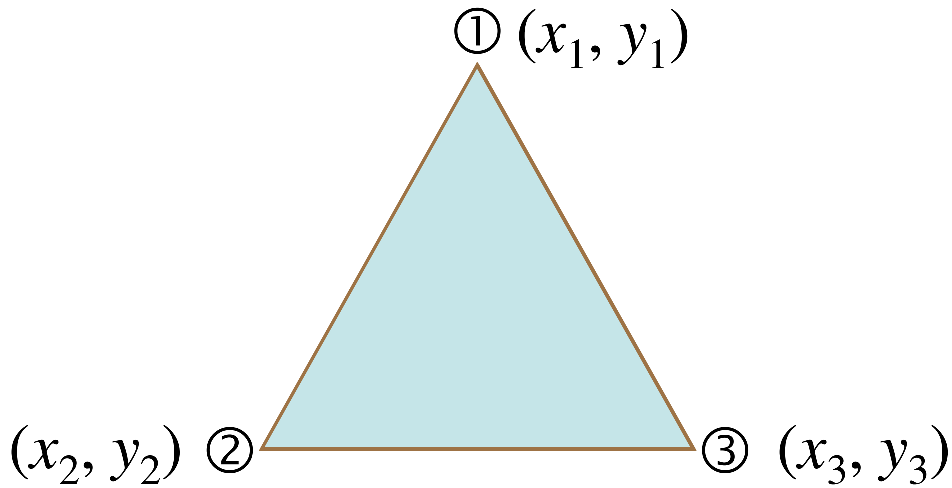
DISCRETIZACION DEL DOMINIO EN ELEMENTOS 2-D



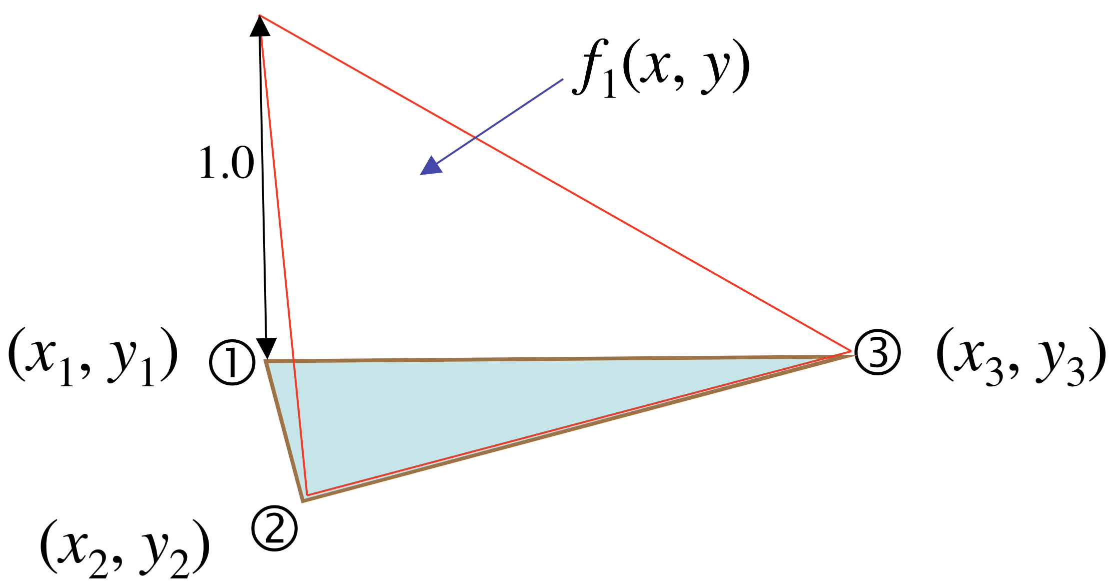
ELEMENTOS TRIANGULARES



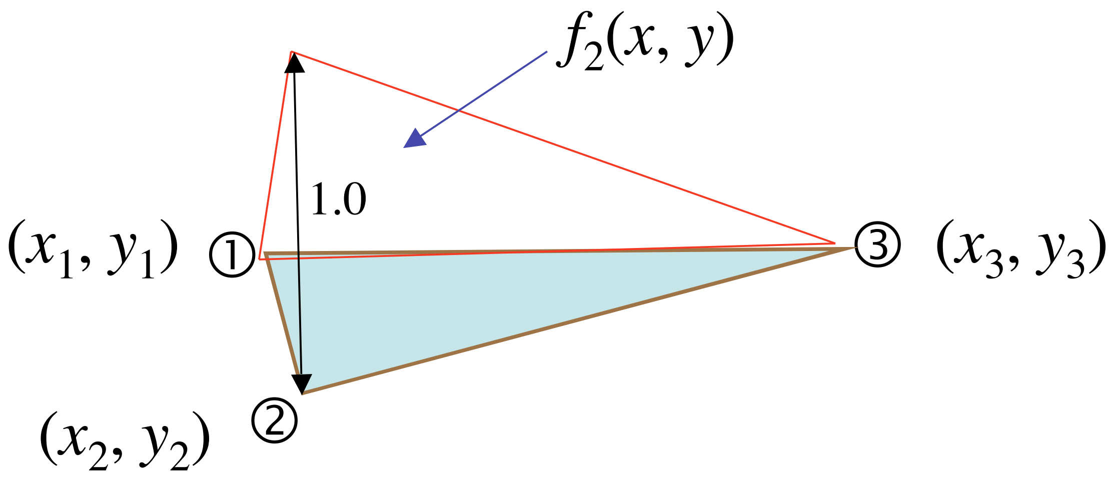
ELEMENTOS TRIANGULARES



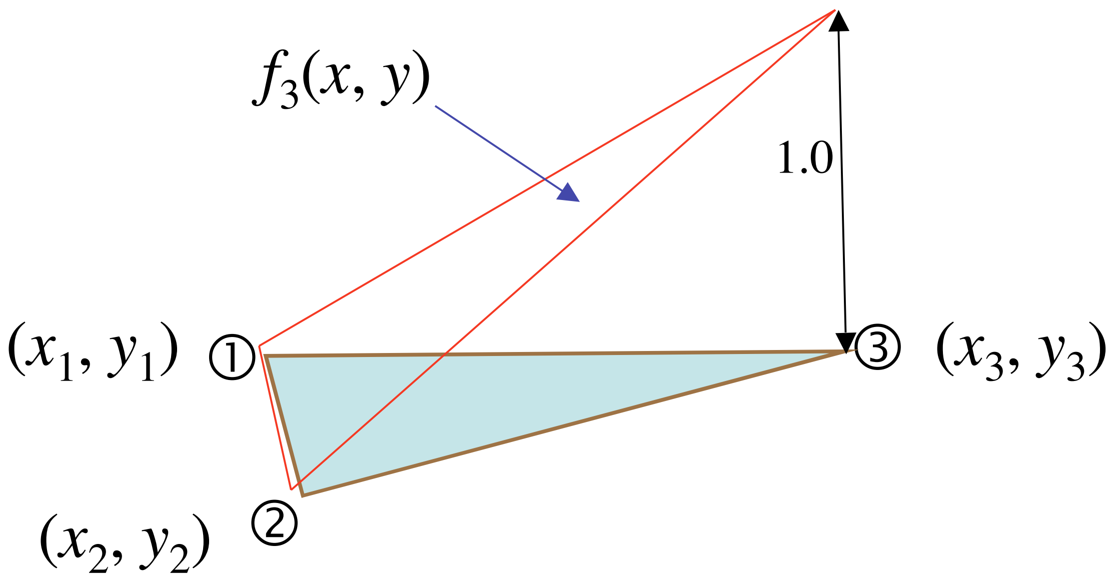
FUNCIONES BASE LINEALES



FUNCIONES BASE LINEALES



FUNCIONES BASE LINEALES



FUNCIONES BASE LINEALES

$$f_i(x, y) = A_i\, x + B_i\, y + C_i$$

$$f_1(x_1, y_1) \;=\; A_1\, x_1 + B_1\, y_1 + \mathrm{C}_1 = 1$$

$$f_1(x_2, y_2) \;=\; A_1\, x_2 + B_1\, y_2 + \mathrm{C}_1 = 0$$

$$f_1(x_3, y_3) \;=\; A_1\, x_3 + B_1\, y_3 + \mathrm{C}_1 = 0$$

FUNCIONES BASE LINEALES

$$f_i(x, y) = A_i\, x + B_i\, y + C_i$$

$$f_2(x_1, y_1) = A_2\, x_1 + B_2\, y_1 + \mathrm{C}_2 = 0$$

$$f_2(x_2, y_2) = A_2\, x_2 + B_2\, y_2 + \mathrm{C}_2 = 1$$

$$f_2(x_3, y_3) = A_2\, x_3 + B_2\, y_3 + \mathrm{C}_2 = 0$$

FUNCIONES BASE LINEALES

$$f_i(x, y) = A_i\, x + B_i\, y + C_i$$

$$f_3(x_1, y_1) \;=\; A_3\, x_1 + B_3\, y_1 + \mathrm{C}_3 = 0$$

$$f_3(x_2, y_2) \;=\; A_3\, x_2 + B_3\, y_2 + \mathrm{C}_3 = 0$$

$$f_3(x_3, y_3) \;=\; A_3\, x_3 + B_3\, y_3 + \mathrm{C}_3 = 1$$

FUNCIONES BASE LINEALES

$$\tilde{U}(x, y) = f_1(x, y)\, u_1 + f_2(x, y)\, u_2 + f_3(x, y)\, u_3$$

$$\tilde{U}(x, y) = \square_1 + \square_2\, x + \square_3\, y$$

$$u_1 = \square_1 + \square_2\, x_1 + \square_3\, y_1$$

$$u_2 = \square_1 + \square_2\, x_2 + \square_3\, y_2$$

$$u_3 = \square_1 + \square_2\, x_3 + \square_3\, y_3$$

FUNCIONES BASE LINEALES

$$f_1 = \frac{(x_2\ y_3 - x_3\ y_2) + (y_2 - y_3)\ x + (x_3 - x_2)\ y}{x_3\ y_2 - x_2\ y_3 - x_1\ y_2 + x_1\ y_3 + x_2\ y_1 - x_3\ y_1}$$

$$f_2 = \frac{(x_3\ y_1 - x_1\ y_3) + (y_3 - y_1)\ x + (x_1 - x_3)\ y}{x_3\ y_2 - x_2\ y_3 - x_1\ y_2 + x_1\ y_3 + x_2\ y_1 - x_3\ y_1}$$

$$f_3 = \frac{(x_1\ y_2 - x_2\ y_1) + (y_1 - y_2)\ x + (x_2 - x_1)\ y}{x_3\ y_2 - x_2\ y_3 - x_1\ y_2 + x_1\ y_3 + x_2\ y_1 - x_3\ y_1}$$

FUNCIONES BASE LINEALES

$$f_i = \frac{1}{2\Delta}(a_i + b_i\,x + c_i\,y)$$

$$a_i = x_j\,y_m - x_m\,y_j$$

$$b_i = y_j - y_m$$

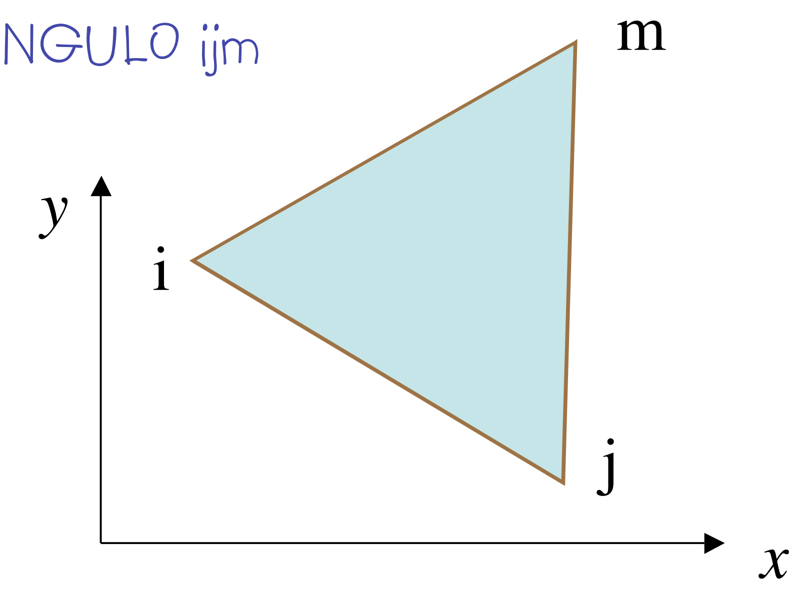
$$c_i = x_m - x_j$$

FUNCIONES BASE LINEALES

$$2\Delta = \det \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix}$$

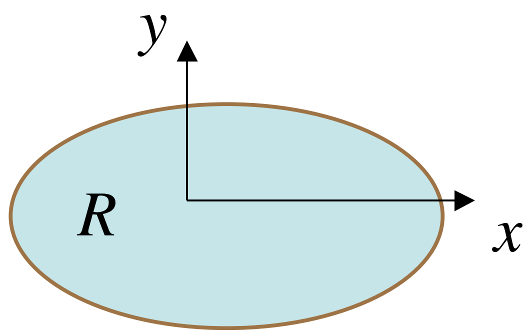
$$2\Delta = x_3\, y_2 - x_2\, y_3 - x_1\, y_2 + x_1\, y_3 + x_2\, y_1 - x_3\, y_1$$

$= 2\,\Delta$ AREA TRIANGULO ijm



EC. DIFUSION 2-D

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(k_x \frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial U}{\partial y}\right)$$



$$\tilde{U}(x, y; t) = \sum_j u_j(t)\, f_j(x, y)$$

$$\frac{\partial \tilde{U}}{\partial t} \square \frac{\partial}{\partial x}\left(k_x \frac{\partial \tilde{U}}{\partial x}\right) \square \frac{\partial}{\partial y}\left(k_y \frac{\partial \tilde{U}}{\partial y}\right) = \square$$

EC. DIFUSION 2-D

$$\int_R W_i \left(\frac{\partial \tilde{U}}{\partial t} - \frac{\partial}{\partial x}\left(\kappa_x \frac{\partial \tilde{U}}{\partial x}\right) - \frac{\partial}{\partial y}\left(\kappa_y \frac{\partial \tilde{U}}{\partial y}\right)\right) dR = 0 \qquad i = 1, \ldots, M$$

$$\tilde{U} = \sum_j u_j \, f_j$$

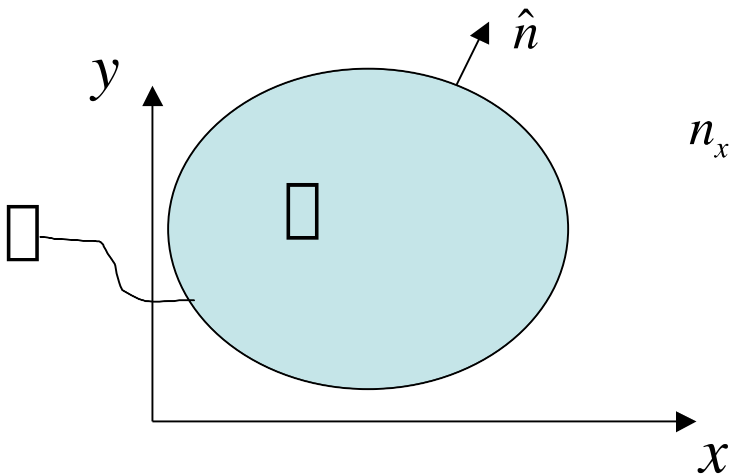
$$\Rightarrow \quad \sum_j \left[\int_R W_i \, f_j \, dR \right] \frac{du_j}{dt} - \sum_j \left[\int_R W_i \, k_{xj} \frac{\partial^2 f_j}{\partial x^2} \, dR + \int_R W_i \, k_{yj} \frac{\partial^2 f_j}{\partial y^2} \, dR \right] u_j = 0$$

SISTEMA DE EDOS:

$$\sum_{\text{nodos internos}} \left(C_{ij}\, u_j + M_{ij} \frac{du_j}{dt} \right) = 0$$

TEOREMA DE GREEN
(INTEGRACION x PARTES 2D):

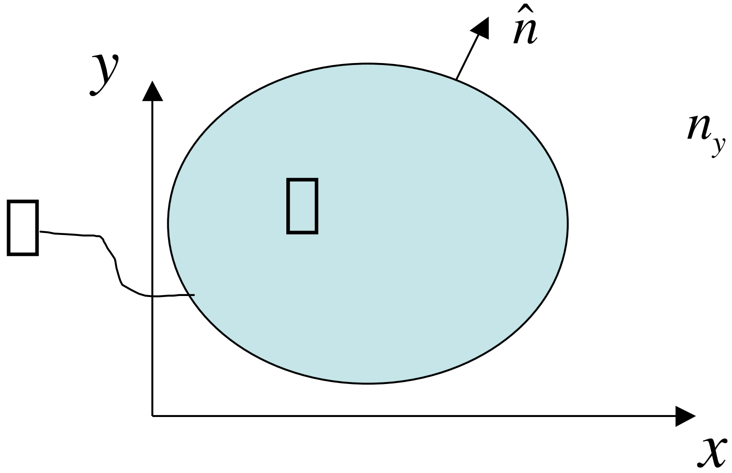
$$\iint_{\Omega} \phi \frac{\partial \psi}{\partial x}\, dx\, dy = -\iint_{\Omega} \frac{\partial \phi}{\partial x} \psi \, dx\, dy + \oint_{\Gamma} \phi \psi \, n_x \, d\Gamma$$



$$n_x = \hat{n} \cdot \hat{i}$$

TEOREMA DE GREEN
(INTEGRACION x PARTES 2D):

$$\square\square_{\square}\ \square\frac{\partial\square}{\partial y}\,dx\,dy=\square\square\square_{\square}\frac{\partial\square}{\partial y}\square\,dx\,dy+\oint_{\square}\square\square\,n_y\,d\square$$



$$n_y=\hat{n}\cdot\hat{j}$$

PARA LOS NODOS INTERIORES:

$$\sum_j \left[\int_R W_i \, f_j \, dR \right] \frac{du_j}{dt} - \sum_j \left[\int_R W_i \, k_{xj} \, \frac{\partial^2 f_j}{\partial x^2} \, dR + \int_R W_i \, k_{yj} \, \frac{\partial^2 f_j}{\partial y^2} \, dR \right] u_j = 0$$

$$\Rightarrow \quad \sum_j \left[\int_R W_i \, f_j \, dx \, dy \right] \frac{du_j}{dt} + \sum_j \left[\int_R k_{xj} \, \frac{\partial W_i}{\partial x} \, \frac{\partial f_j}{\partial x} \, dx \, dy + \int_R k_{yj} \, \frac{\partial W_i}{\partial y} \, \frac{\partial f_j}{\partial y} \, dx \, dy \right] u_j = 0$$

GALERKIN: $\quad W_i = f_i$

$$\sum_j \left[\iint_R f_i\, f_j \, dx\, dy\right]\frac{du_j}{dt} + \sum_j \left[\iint_R k_{xj}\,\frac{\partial f_i}{\partial x}\frac{\partial f_j}{\partial x}\, dx\, dy + \iint_R k_{yj}\,\frac{\partial f_i}{\partial y}\frac{\partial f_j}{\partial y}\, dx\, dy\right] u_j = 0$$

PERO:

$$f_i = \frac{1}{2\Delta}\left(a_i + b_i\, x + c_i\, y\right)$$

$$\frac{\partial f_i}{\partial x} = \frac{b_i}{2\Delta}$$

$$\frac{\partial f_i}{\partial y} = \frac{c_i}{2\Delta}$$

$$\frac{\partial f_i}{\partial x}\frac{\partial f_j}{\partial x}=\frac{b_i\, b_j}{(2\,\square)^2} \qquad \frac{\partial f_i}{\partial y}\frac{\partial f_j}{\partial y}=\frac{c_i\, c_j}{(2\,\square)^2}$$

$$f_i\, f_j=\frac{1}{(2\,\square)^2}(a_i+b_i\, x+c_i\, y)(a_j+b_j\, x+c_j\, y)$$

$$f_i\, f_j=\frac{1}{(2\,\square)^2}(a_i\, a_j+(a_i\, b_j+b_i\, a_j)\, x+(a_i\, c_j+c_i\, a_j)\, y+$$
$$(b_i\, c_j+c_i\, b_j)\, x\, y+b_i\, b_j\, x^2+c_i\, c_j\, y^2)$$

PARA CADA ELEMENTO:

$$\sum_j \left[\int_R f_i\, f_j\, dx\, dy\right]\frac{du_j}{dt} + \sum_j \left[\int_R k_{xj}\, \frac{\partial f_i}{\partial x}\,\frac{\partial f_j}{\partial x}\, dx\, dy + \int_R k_{yj}\, \frac{\partial f_i}{\partial y}\,\frac{\partial f_j}{\partial y}\, dx\, dy\right] u_j = 0$$

$$\Rightarrow \quad \int_\Omega k_{xj}\, \frac{\partial f_i}{\partial x}\,\frac{\partial f_j}{\partial x}\, dx\, dy = k_{xj} \int_\Omega \frac{b_i\, b_j}{(2\Delta)^2}\, dx\, dy = k_{xj}\, \frac{b_i\, b_j}{(2\Delta)^2}\, \underbrace{\int_\Omega dx\, dy}_{\Delta}$$

$$\Rightarrow \quad \int_\Omega k_{xj}\, \frac{\partial f_i}{\partial x}\,\frac{\partial f_j}{\partial x}\, dx\, dy = k_{xj}\, \frac{b_i\, b_j}{4\Delta}$$

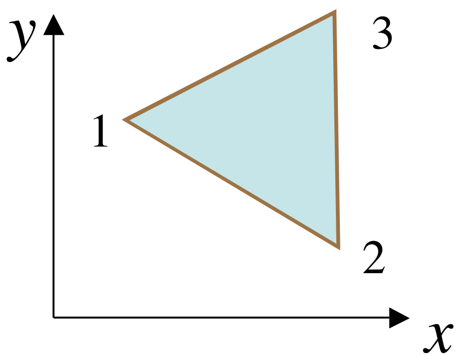
$$\Rightarrow \quad \int_\Omega k_{yj}\, \frac{\partial f_i}{\partial y}\,\frac{\partial f_j}{\partial y}\, dx\, dy = k_{yj}\, \frac{c_i\, c_j}{4\Delta}$$

PARA EL CASO PERMANENTE, COEFICIENTES CONSTANTES:

$$\sum_{j} \left[\underbrace{\int_{R} k_{xj} \frac{\partial f_i}{\partial x} \frac{\partial f_j}{\partial x}\, dx\, dy}_{k_x \frac{b_i b_j}{4\Delta}} + \underbrace{\int_{R} k_{yj} \frac{\partial f_i}{\partial y} \frac{\partial f_j}{\partial y}\, dx\, dy}_{k_y \frac{c_i c_j}{4\Delta}} \right] u_j = 0$$

MATRICES ELEMENTALES:

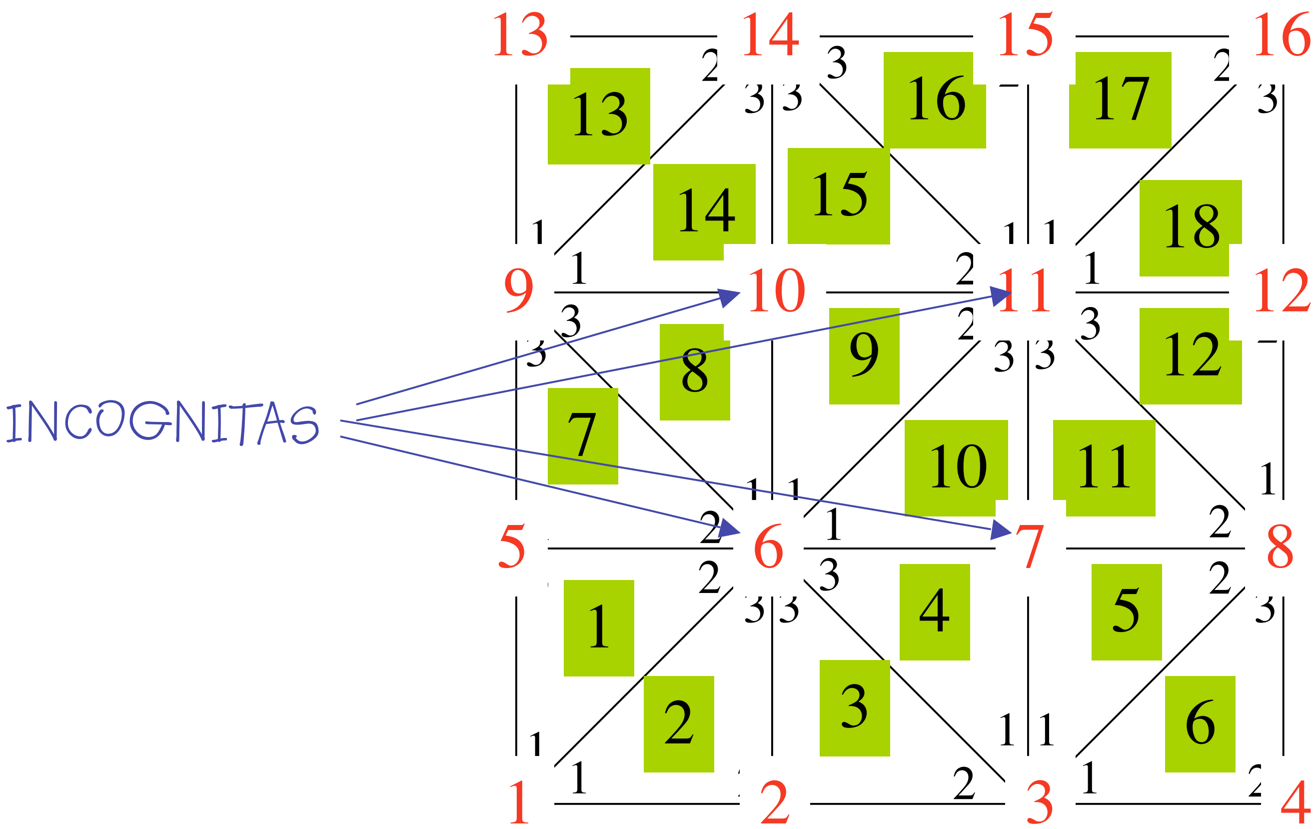
$$\sum_j \left[\underbrace{\int_R k_{xj} \frac{\partial f_i}{\partial x} \frac{\partial f_j}{\partial x}\, dx\, dy}_{k_x \frac{b_i b_j}{4\Delta}} + \underbrace{\int_R k_{yj} \frac{\partial f_i}{\partial y} \frac{\partial f_j}{\partial y}\, dx\, dy}_{k_y \frac{c_i c_j}{4\Delta}} \right] u_j = 0$$

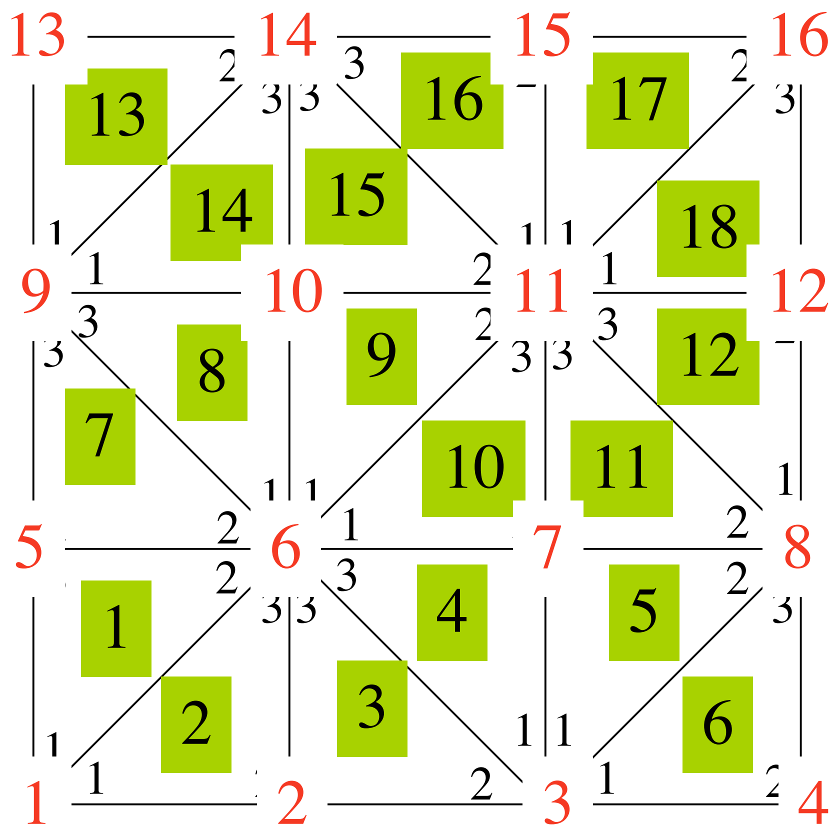


MATRICES ELEMENTALES:

$$\frac{1}{4\,\square}\left\{ k_x \begin{vmatrix} b_1\,b_1 & b_1\,b_2 & b_1\,b_3 \\ b_2\,b_1 & b_2\,b_2 & b_2\,b_3 \\ b_3\,b_1 & b_3\,b_2 & b_3\,b_3 \end{vmatrix} + k_y \begin{vmatrix} c_1\,c_1 & c_1\,c_2 & c_1\,c_3 \\ c_2\,c_1 & c_2\,c_2 & c_2\,c_3 \\ c_3\,c_1 & c_3\,c_2 & c_3\,c_3 \end{vmatrix} \right\} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix} = 0$$

EJEMPLO:





$$f_1 u_1 + f_2 u_6 + f_3 u_5 = \tilde{u}_1$$

$$f_1 u_1 + f_2 u_2 + f_3 u_6 = \tilde{u}_2$$

$$f_1 u_2 + f_2 u_3 + f_3 u_6 = \tilde{u}_3$$

$$f_1 u_3 + f_2 u_7 + f_3 u_6 = \tilde{u}_4$$

$$f_1 u_3 + f_2 u_8 + f_3 u_7 = \tilde{u}_5$$

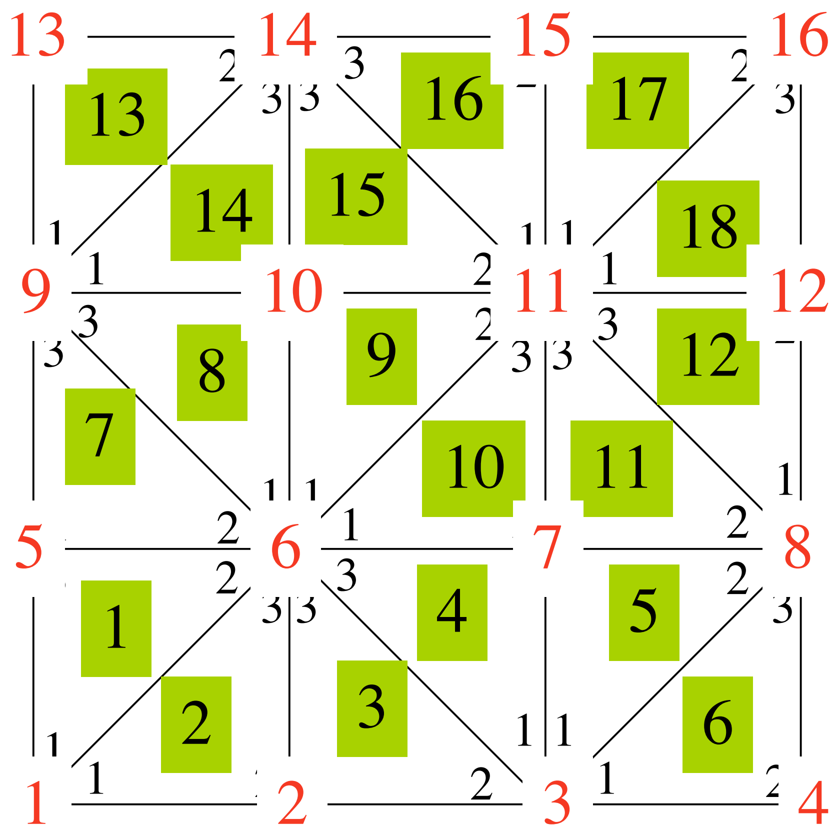
$$f_1 u_5 + f_2 u_6 + f_3 u_9 = \tilde{u}_7$$

$$f_1 u_6 + f_2 u_{10} + f_3 u_9 = \tilde{u}_8$$

$$f_1 u_6 + f_2 u_{11} + f_3 u_{10} = \tilde{u}_9$$

$$f_1 u_7 + f_2 u_8 + f_3 u_{11} = \tilde{u}_{11}$$

$$f_1 u_8 + f_2 u_{12} + f_3 u_{11} = \tilde{u}_{12}$$



$$f_1\, u_9 + f_2\, u_{10} + f_3\, u_{14} = \tilde{u}_{14}$$

$$f_1\, u_{10} + f_2\, u_{11} + f_3\, u_{14} = \tilde{u}_{15}$$

$$f_1\, u_{11} + f_2\, u_{15} + f_3\, u_{14} = \tilde{u}_{16}$$

$$f_1\, u_{11} + f_2\, u_{16} + f_3\, u_{15} = \tilde{u}_{17}$$

$$f_1\, u_{11} + f_2\, u_{12} + f_3\, u_{16} = \tilde{u}_{18}$$

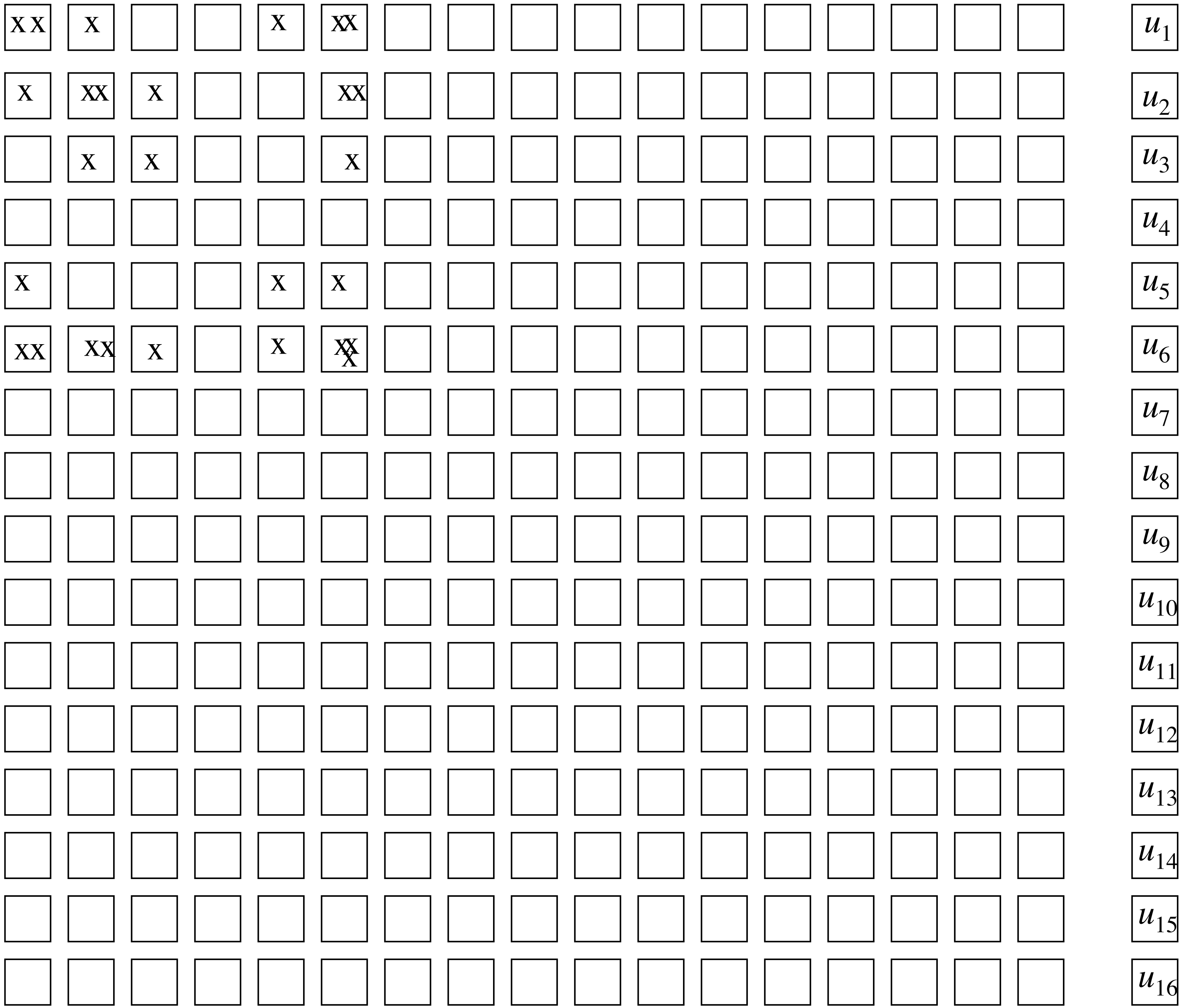
$$\boxed{\begin{matrix} K_{11}^1 & K_{12}^1 & K_{13}^1 \\ K_{21}^1 & K_{22}^1 & K_{23}^1 \\ K_{31}^1 & K_{32}^1 & K_{33}^1 \end{matrix}} \boxed{\begin{matrix} u_1 \\ u_6 \\ u_5 \end{matrix}} \quad \boxed{\begin{matrix} K_{11}^2 & K_{12}^2 & K_{13}^2 \\ K_{21}^2 & K_{22}^2 & K_{23}^2 \\ K_{31}^2 & K_{32}^2 & K_{33}^2 \end{matrix}} \boxed{\begin{matrix} u_1 \\ u_2 \\ u_6 \end{matrix}} \quad \boxed{\begin{matrix} K_{11}^3 & K_{12}^3 & K_{13}^3 \\ K_{21}^3 & K_{22}^3 & K_{23}^3 \\ K_{31}^3 & K_{32}^3 & K_{33}^3 \end{matrix}} \boxed{\begin{matrix} u_2 \\ u_3 \\ u_6 \end{matrix}} \quad \boxed{\begin{matrix} K_{11}^4 & K_{12}^4 & K_{13}^4 \\ K_{21}^4 & K_{22}^4 & K_{23}^4 \\ K_{31}^4 & K_{32}^4 & K_{33}^4 \end{matrix}} \boxed{\begin{matrix} u_3 \\ u_7 \\ u_6 \end{matrix}}$$

$$\boxed{\begin{matrix} K_{11}^5 & K_{12}^5 & K_{13}^5 \\ K_{21}^5 & K_{22}^5 & K_{23}^5 \\ K_{31}^5 & K_{32}^5 & K_{33}^5 \end{matrix}} \boxed{\begin{matrix} u_3 \\ u_8 \\ u_7 \end{matrix}} \quad \boxed{\begin{matrix} K_{11}^6 & K_{12}^6 & K_{13}^6 \\ K_{21}^6 & K_{22}^6 & K_{23}^6 \\ K_{31}^6 & K_{32}^6 & K_{33}^6 \end{matrix}} \boxed{\begin{matrix} u_3 \\ u_4 \\ u_8 \end{matrix}} \quad \boxed{\begin{matrix} K_{11}^7 & K_{12}^7 & K_{13}^7 \\ K_{21}^7 & K_{22}^7 & K_{23}^7 \\ K_{31}^7 & K_{32}^7 & K_{33}^7 \end{matrix}} \boxed{\begin{matrix} u_5 \\ u_6 \\ u_9 \end{matrix}} \quad \boxed{\begin{matrix} K_{11}^8 & K_{12}^8 & K_{13}^8 \\ K_{21}^8 & K_{22}^8 & K_{23}^8 \\ K_{31}^7 & K_{32}^7 & K_{33}^7 \end{matrix}} \boxed{\begin{matrix} u_6 \\ u_{10} \\ u_9 \end{matrix}}$$

$$\boxed{\begin{matrix} K_{11}^9 & K_{12}^9 & K_{13}^9 \\ K_{21}^9 & K_{22}^9 & K_{23}^9 \\ K_{31}^9 & K_{32}^9 & K_{33}^9 \end{matrix}} \boxed{\begin{matrix} u_6 \\ u_{11} \\ u_{10} \end{matrix}} \quad \boxed{\quad 10 \quad} \boxed{\begin{matrix} u_6 \\ u_7 \\ u_{11} \end{matrix}} \quad \boxed{\quad 11 \quad} \boxed{\begin{matrix} u_7 \\ u_8 \\ u_{11} \end{matrix}} \quad \boxed{\quad 12 \quad} \boxed{\begin{matrix} u_8 \\ u_{12} \\ u_{11} \end{matrix}}$$

$$\boxed{\quad 13 \quad} \boxed{\begin{matrix} u_9 \\ u_{14} \\ u_{13} \end{matrix}} \quad \boxed{\quad 14 \quad} \boxed{\begin{matrix} u_9 \\ u_{10} \\ u_{14} \end{matrix}} \quad \boxed{\quad 15 \quad} \boxed{\begin{matrix} u_{10} \\ u_{11} \\ u_{14} \end{matrix}} \quad \boxed{\quad 16 \quad} \boxed{\begin{matrix} u_{11} \\ u_{15} \\ u_{14} \end{matrix}}$$

$$\boxed{\quad 17 \quad} \boxed{\begin{matrix} u_{11} \\ u_{16} \\ u_{15} \end{matrix}} \quad \boxed{\quad 18 \quad} \boxed{\begin{matrix} u_{11} \\ u_{12} \\ u_{16} \end{matrix}}$$


u_1
u_2
u_3
u_4
u_5
u_6
u_7
u_8
u_9
u_{10}
u_{11}
u_{12}
u_{13}
u_{14}
u_{15}
u_{16}

$$\begin{vmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{vmatrix} \begin{vmatrix} u_6 \\ u_7 \\ u_{10} \\ u_{11} \end{vmatrix} = \text{B.C.}$$