

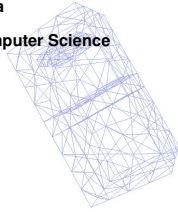
TERMINAL-EDGE ALGORITHMS FOR TRIANGULATIONS

An integrated approach for mesh generation

Oberwolfach, March 2004

María-Cecilia Rivara

Department of Computer Science
University of Chile



Rivara Oberwolfach 2004

1

Outline

- Motivation
- Triangulation problems
 - point set triangulation
 - polygon triangulation
 - mesh improvement
 - automatic mesh generation
 - mesh refinement / derefinement
 - parallel algorithms

- Basic concepts and previous algorithms
 - Delaunay algorithms
 - Longest edge algorithms
- Terminal-edges (longest-edge) algorithms
 - concepts
 - properties
 - algorithms

- Conclusions and future research

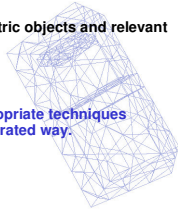
Rivara Oberwolfach 2004

2

Motivation

- Finite element applications: engineering and scientific applications → quality triangulations are needed to assure convergence
- Adaptive refinement / derefinement of the mesh through the finite element computations
- 2d and 3d object approximation
- A good representation of geometric objects and relevant details.
- Control volume methods

Terminal-Edge Algorithms are appropriate techniques to deal with these issues in an integrated way.

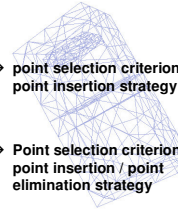


Rivara Oberwolfach 2004

3

Triangulation problems and algorithms

- Point set triangulation in 2 and 3-dimensions → Delaunay triangulation, constrained Delaunay triangulation (CDT)
- Polygon (PSLG) triangulation
- 3D objects triangulation
- Automatic mesh generation → point selection criterion + point insertion strategy
- Mesh improvement
- Mesh refinement → Point selection criterion + point insertion / point elimination strategy
- Mesh derefinement
- Parallel mesh generation

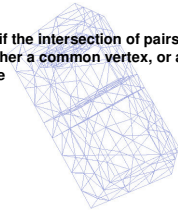
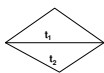


Rivara Oberwolfach 2004

4

Basic concepts: valid triangulations

- In 2D, T is a valid triangulation if the intersection of pairs of adjacent triangles t_1, t_2 in T is either a common vertex or a common edge.
- In 3D, T is valid 3D triangulation if the intersection of pairs of adjacent tetrahedra t_1, t_2 is either a common vertex, or a common edge, or a common face

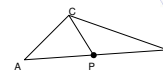


Rivara Oberwolfach 2004

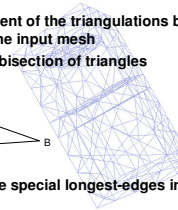
5

Basic concepts: longest-edge refinement algorithms

- Previous longest edge refinement algorithms (Rivara 84, 87, 92)
 - developed for adaptive fem
 - they allow the local refinement of the triangulations by maintaining the quality of the input mesh
 - based on the longest-edge bisection of triangles



Remark: Terminal-edges are special longest-edges in the triangulation

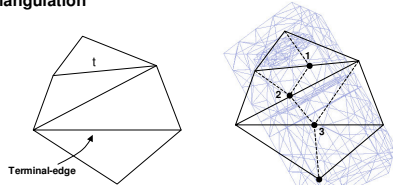


Rivara Oberwolfach 2004

6

Longest-edge refinement algorithm for adaptive finite element methods

- Given a target triangle t , this triangle and a set of its neighbors are refined in order to maintain a valid triangulation



Rivara Oberwolfach 2004

7

Properties

- The iterative longest-edge bisection of a triangle t_0 assures that $\min \angle(t) \geq \alpha_0/2$, where $\alpha_0 = \min \angle$ of t_0
- A finite set of similar distinct triangles are produced
- Longest-edge refinement algorithms inherit these properties
- Fractal property

THE QUALITY OF THE MESHES DEPEND ON THE QUALITY OF THE INITIAL TRIANGULATION

Rivara Oberwolfach 2004

8

Example: adaptive triangulation obtained with the refinement algorithm

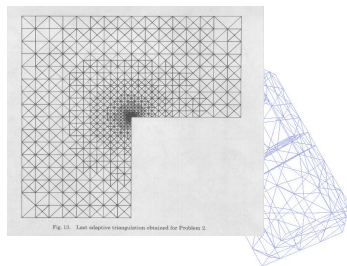


Fig. 10. Adaptive triangulation obtained for Problem 2.

Rivara Oberwolfach 2004

9

Refinement / Derefinement algorithm

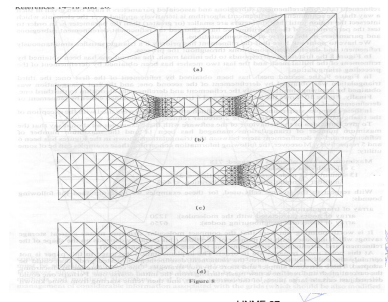


Figure 9

IJNME 97

Rivara Oberwolfach 2004

10

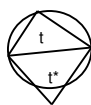
Construction of the initial triangulation Delaunay triangulation (DT)

- Problem: Given a point set P , construct a triangulation of P

$P = \{ \text{points} \}$

T is a DT of P if the circuncircle of every triangle t in T does not include any vertex of P in its interior

t and t^* are locally Delaunay triangles



Rivara Oberwolfach 2004

11

Optimality properties of the Delaunay triangulation 2D

- Constructs the most equilateral triangulation
- Maximizes the minimum angle
- Minimizes the maximum angle
- Maximizes the maximum radius of an enclosed circle
- Others

Bern 97

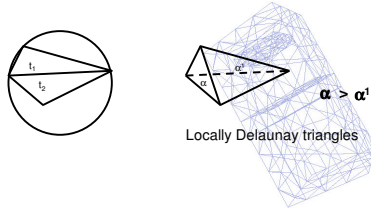
In Handbook of Discrete and Computational Geometry
Goodman and O'Rourke (eds)

Rivara Oberwolfach 2004

12

Important property: diagonal swapping

t, t^* : a pair of non-Delaunay triangles
They are transformed in a pair of locally Delaunay triangles by diagonal swapping
This operation improves the angles by pairs



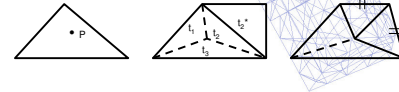
Rivara Oberwolfach 2004

13

An incremental Delaunay algorithm based on edge swapping can be stated

Sketch

- P : point to be inserted
- The triangle that contains P is found
- P is joined with the vertices of P forming triangles t_1, t_2, t_3
- Recursively, if the pairs t_i, t_i^* are not locally Delaunay, edge swapping is performed.



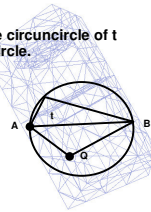
Rivara Oberwolfach 2004

14

Constrained Delaunay triangulation CDT in 2D

- A CDT includes fixed edges to be respected
- A relaxation of the Delaunay triangulation conditions is used
- A kind of visibility problem
- If AB is a constrained edge, then the circuncircle of t doesn't see the point Q inside the circle.

This concept allows the triangulation of polygons (PSLG)



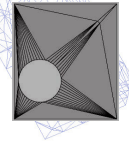
Rivara Oberwolfach 2004

15

Over the geometric quality of the triangulations

- Delaunay algorithm constructs the most equilateral triangulation for the given data
- Constrained Delaunay algorithm constructs the most equilateral triangulation for the given data, considering the constrained edges

The quality of the triangulation depends on the point distribution



Rivara Oberwolfach 2004

16

Automatic Quality triangulation problem

Problem:

Given a 2D or 3D bounded object, construct a quality triangulation of this object

- Additional points need to be inserted in order to improve the point distribution
- A point selection criteria + a point insertion strategy are needed



Rivara Oberwolfach 2004

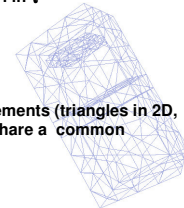
17

Terminal-Edge: key concept for mesh improvement and mesh refinement

- Terminal-Edge l in triangulation τ
Special edge in τ such that l is longest-edge of every element that shares edge l in τ



- Terminal-star: set of elements (triangles in 2D, tetrahedra in 3D) that share a common terminal edge

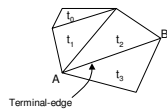


Rivara Oberwolfach 2004

18

Lepp(t): a key concept for finding terminal-edges

- Longest-Edge Propagation Path (t) $\text{Lepp}(t)$



$\text{Lepp}(t_0) = \{t_0, t_1, t_2, t_3\}$

$\text{Lepp}(t_0)$ identifies terminal-edge AB associated to t_0 in τ

Longest-edge (t_i) > longest-edge (t_{i-1})

Improvement / Refinement Algorithms

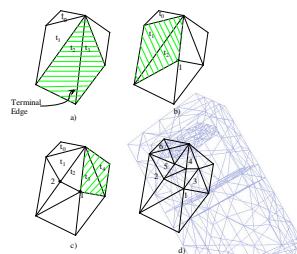
Common basic idea: refinement of Terminal-Edges

- Input: Set S of target elements to be improved or refined in mesh τ
- For each t in S do
 - While t remains in the mesh do
 - Find $\text{Lepp}(t)$ and associated terminal-edge
 - Select point P to be inserted in τ as the midpoint of a terminal-edge
 - Insert point P in mesh (*)

(*) Refinement \Rightarrow longest-edge bisection

(*) Improvement \Rightarrow Delaunay point insertion

Terminal Edge (Terminal Star) Refinement Algorithm



Algorithm maintains mesh quality!

Terminal star refinement algorithms: improved longest-edge algorithms

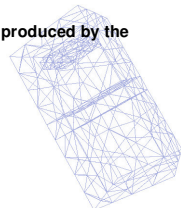
- Terminal-star refinement operation: a very local operation, 2D and 3D
- We work on the parallelization of terminal-star refinement algorithms for 3D meshes

(with N. Chrisochoides)

- Algorithms very appropriate for the self-adaptive solution of PDE, 2D and 3D

Derefinement Algorithm

- Based on the derefinement of previous terminal-edges
- It applies over triangulations produced by the refinement algorithm

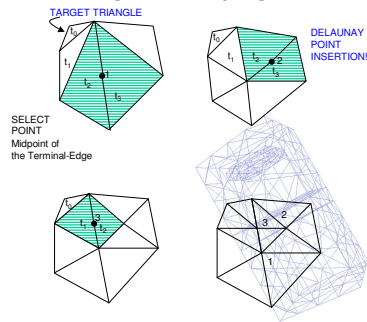


Improvement of triangulations and automatic construction of quality triangulations

- Terminal-edges associated to bad quality elements (triangles, tetrahedra) are the best places for point insertion
- $\text{Lepp}(t)$: for bad quality triangle t , allows to find associated terminal-edges



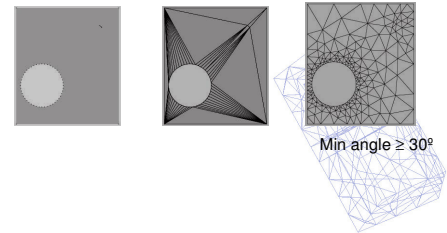
Triangle improvement terminal-edge Delaunay algorithm



Rivara Oberwolfach 2004

25

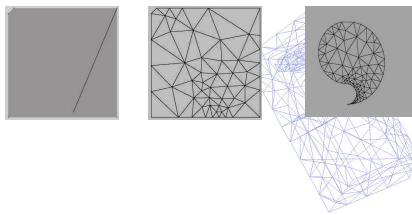
Automatic quality triangulation algorithm



Rivara Oberwolfach 2004

26

More examples



Rivara Oberwolfach 2004

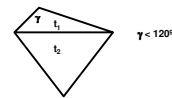
27

Properties of the Mesh Improvement Algorithm

- Works over Delaunay Meshes
- Performs Delaunay insertion of midpoints of terminal-edges
- Produces triangulations with $\min \text{angle} \geq 30^\circ$

- If (t_1, t_2) are terminal triangles (sharing a terminal-edge) then

target angle $(t_1) < 120^\circ$
target angle $(t_2) < 120^\circ$

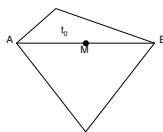


Rivara Oberwolfach 2004

28

Improvement Property

AB: terminal-edge
 t_0 : bad-quality triangle



- Delaunay point insertion of M
- produces an improved triangle t_1
 - bad quality triangle t_2 is destroyed (or will be destroyed) by edge-swapping
 - If angle $CMB > 120^\circ$ then BC can not be a terminal edge

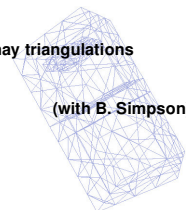
Rivara Oberwolfach 2004

29

Research in course in this direction

- Algorithm tuning and theoretical results on optimal quality triangulation
- Improvement of non-Delaunay triangulations

(with B. Simpson)



Rivara Oberwolfach 2004

30

Applications of terminal-edge concepts

2-dimensional software

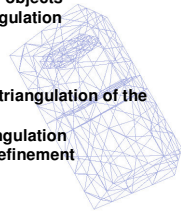
- Automatic quality triangulation
- Mesh Refinement / Derefinement

• Surface triangulation of 3D objects

- Automatic quality triangulation
- Mesh Refinement

• Volume triangulation

- Constrained Delaunay triangulation of the surface points
- Improved Volume Triangulation
- Mesh Refinement / Derefinement

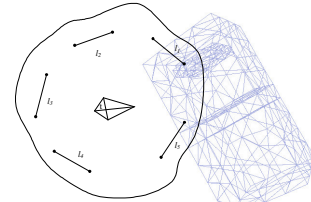


Rivara Oberwolfach 2004

31

Lepp(t) in 3-dimensions

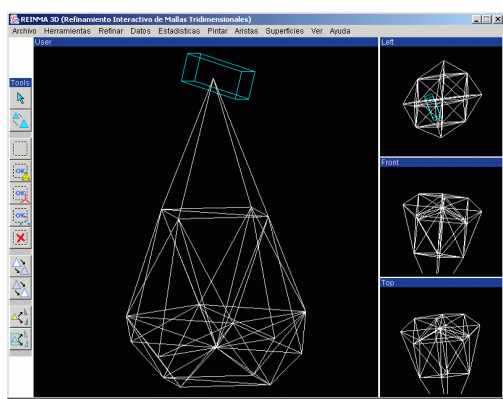
- A submesh associated to an individual element t
- Allows the identification of a set of terminal-edges



- Each Terminal-edge is shared by a set of tetrahedra

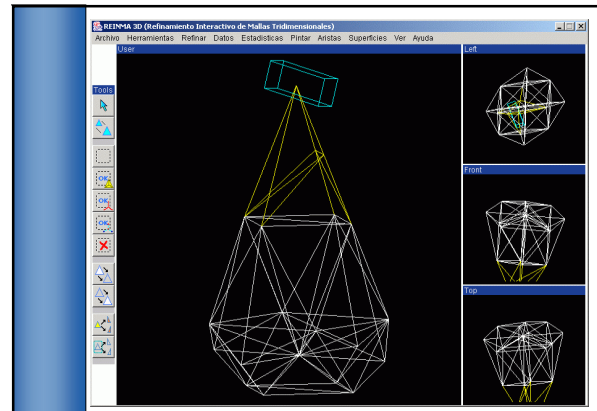
Rivara Oberwolfach 2004

32



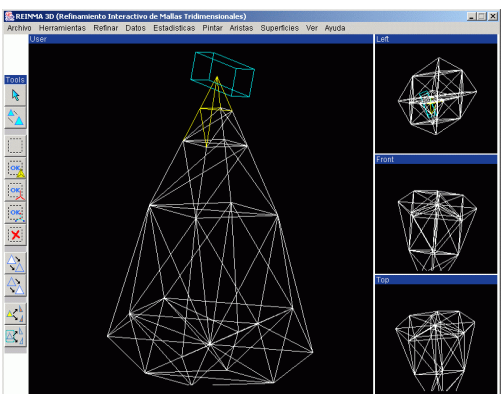
Rivara Oberwolfach 2004

33



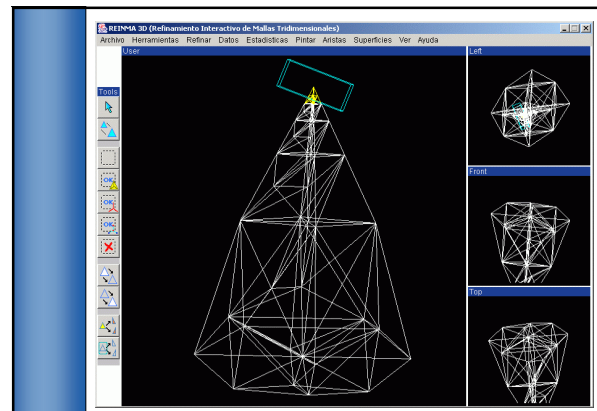
Rivara Oberwolfach 2004

34



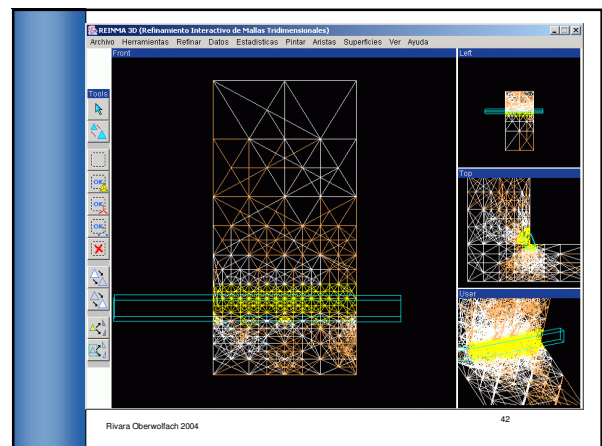
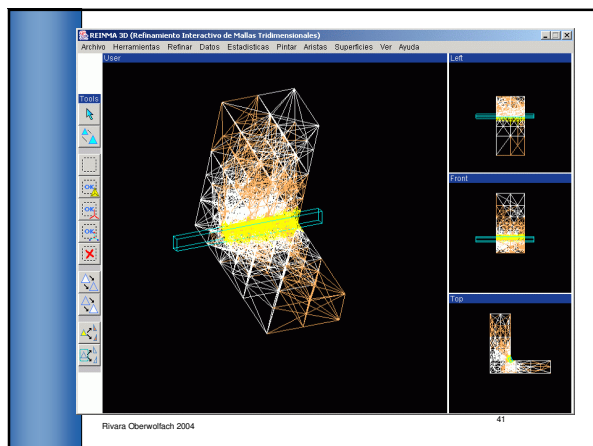
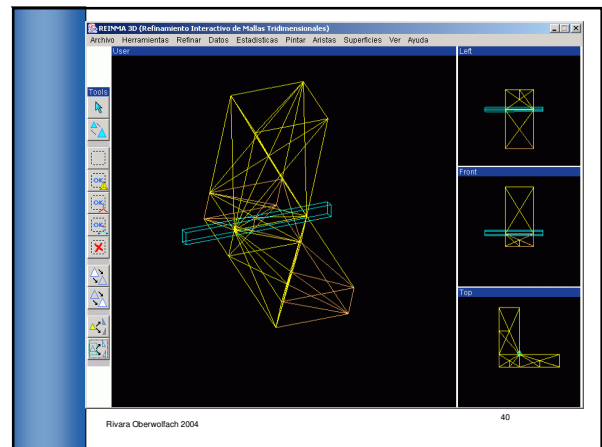
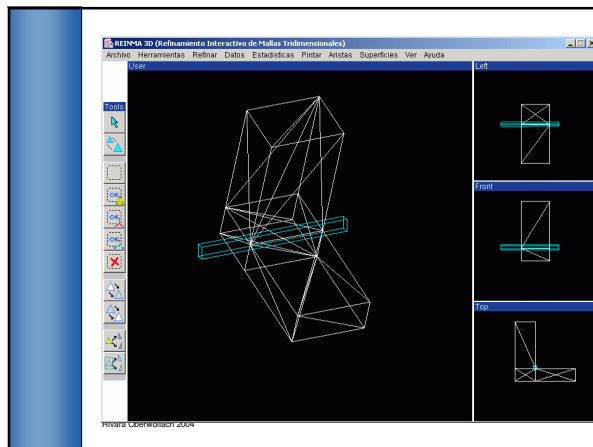
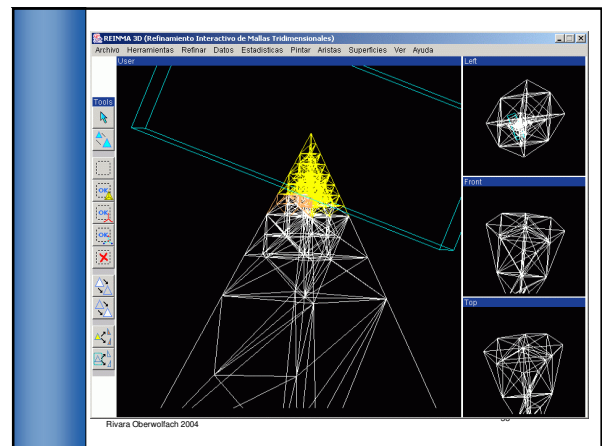
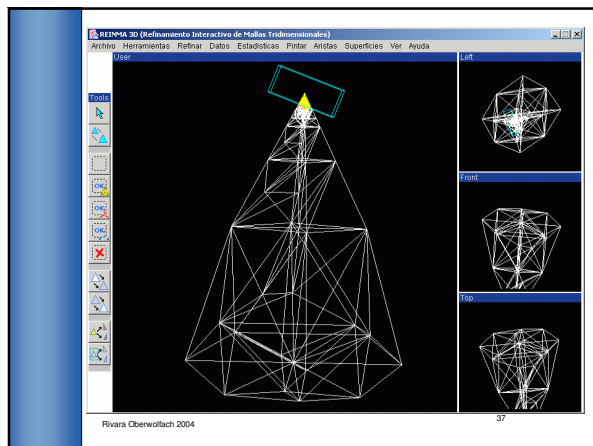
Rivara Oberwolfach 2004

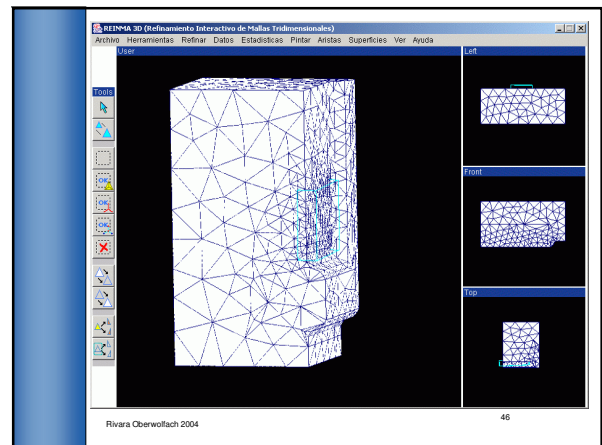
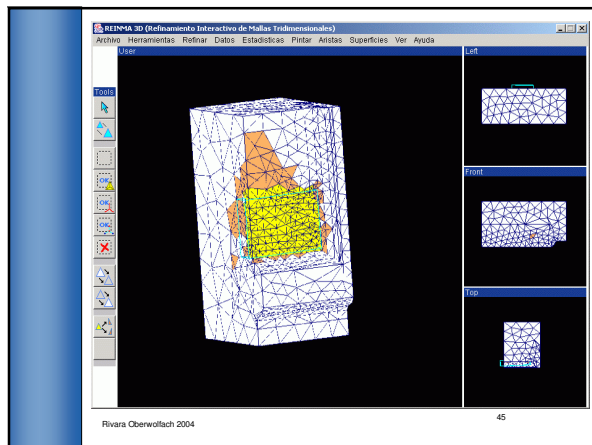
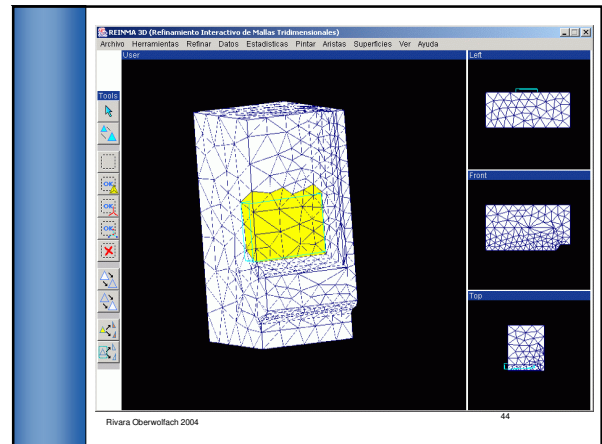
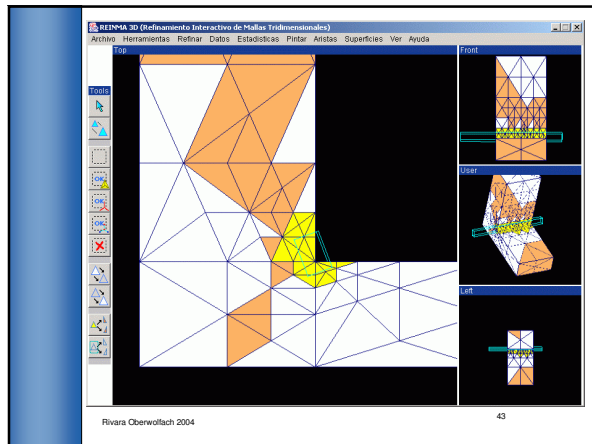
35



Rivara Oberwolfach 2004

36





Conclusions

- Integrated and robust approach for automatic and adaptive (2D, surface, volume) mesh generation
 - Constrained Delaunay algorithm
 - Terminal-edge based algorithms
 - Terminal star based algorithms
- Research in course
 - Theoretical results on optimal triangulations and algorithm tuning
 - 3D meshing software
 - Parallel algorithms

