

Examen MA22A-01/07-12-02

1.-

a) Como la densidad es constante e igual a 1, numéricamente la Masa y el Volumen coinciden.

$$Vol = \int_0^{\frac{\pi}{2}} \int_0^3 \int_0^{r \sin \theta} r dz dr d\theta = 9$$

$$\bar{x} = \frac{1}{9} \int_0^{\frac{\pi}{2}} \int_0^3 \int_0^{r \sin \theta} r \cos \theta dz dr d\theta = \frac{9}{8}$$

$$\bar{y} = \frac{1}{9} \int_0^{\frac{\pi}{2}} \int_0^3 \int_0^{r \sin \theta} r \sin \theta dz dr d\theta = \frac{9\pi}{16}$$

$$\bar{z} = \frac{1}{9} \int_0^{\frac{\pi}{2}} \int_0^3 \int_0^{r \sin \theta} z r dz dr d\theta = \frac{9\pi}{32}$$

b)

$$i) \int_0^{\pi} \int_0^1 (r \sin \theta) r dr d\theta = \frac{2}{3}$$

$$ii) \int_0^{\pi} \int_0^1 (r \cos \theta) (r \sin \theta) r dr d\theta = 0$$

$$iii) \int_0^{\pi} \int_0^1 (r \sin \theta)^2 r dr d\theta = \frac{\pi}{8}$$

$$iv) \int_0^{\pi} \int_0^1 (r \cos \theta)^2 (r \sin \theta) r dr d\theta = \frac{2}{15} \quad v) \int_0^{\pi} \int_0^1 r^2 (r \sin \theta) r dr d\theta = \frac{2}{5}$$

2.-

$$a) \text{ PDQ } Vol = \frac{AB}{4} (2aB + 2bA + 4c)$$

Demostración

$$Vol = \int_0^B \int_0^A \int_0^{ax+by+c} dz dy dx = \int_0^B (aAx + \frac{bA^2}{2} + cA) dx = \frac{AB}{4} (2aB + 2bA + 4c)$$

b) El problema a resolver es:

$$f(x) = \|x\|_2^2 = x^t x$$

$$sa \quad q(x) = x^t A x = 1$$

$$\Rightarrow L(x, \lambda) = x^t x - \lambda(x^t A x - 1)$$

Derivando se obtiene $2x - \lambda 2Ax = 0 \Rightarrow Ax = \frac{1}{\lambda} x$, por lo tanto $1/\lambda$ es un valor propio de la matriz A, como A es definida positiva, todos sus valores propios son reales y positivos.

Como $f(x) = x^t x = x^t \lambda A x = \lambda x^t A x = \lambda$ luego el mínimo de f corresponderá al recíproco del máximo valor propio y el máximo de f será el recíproco del mínimo valor propio.

Para el caso de $q(x, y, z) = 7x^2 + 4y^2 + 4z^2 + 4xy - 4xz - 2yz$, la matriz A será

$$A = \begin{bmatrix} 7 & 2 & -2 \\ 2 & 4 & -1 \\ -2 & -1 & 4 \end{bmatrix}$$

Los valores propios de A son 3 y 9. por lo tanto

$$f_{\min} = \frac{1}{9} \quad f_{\max} = \frac{1}{3}$$

3.- $f(x, y) = g(xy, \frac{x}{y})$ dos veces diferenciable.

PDQ

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \Leftrightarrow (uv + \frac{u}{v}) \frac{\partial^2 g}{\partial u^2} + 2(1 - v^2) \frac{\partial^2 g}{\partial u \partial v} + \frac{v}{u}(1 + v^2) \frac{\partial^2 g}{\partial v^2} + 2 \frac{v^2}{u} \frac{\partial g}{\partial v} = 0$$

$$\left. \begin{array}{l} u = xy \\ v = \frac{x}{y} \end{array} \right\} \quad \begin{array}{l} uv = x^2 \Rightarrow x = \sqrt{uv} \\ \frac{u}{v} = y^2 \Rightarrow y = \sqrt{\frac{u}{v}} \end{array}$$

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial g}{\partial v} \frac{\partial v}{\partial x}$$

$$\frac{\partial^2 f}{\partial x^2} = \left(\frac{\partial^2 g}{\partial u^2} \frac{\partial u}{\partial x} + \frac{\partial^2 g}{\partial v \partial u} \frac{\partial v}{\partial x} \right) \frac{\partial u}{\partial x} + \frac{\partial g}{\partial u} \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 g}{\partial u \partial v} \frac{\partial u}{\partial x} + \frac{\partial^2 g}{\partial v^2} \frac{\partial v}{\partial x} \right) \frac{\partial v}{\partial x} + \frac{\partial g}{\partial v} \frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 g}{\partial u^2} \left(\frac{\partial u}{\partial x} \right)^2 + 2 \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \right) \frac{\partial^2 g}{\partial u \partial v} + \frac{\partial^2 g}{\partial v^2} \left(\frac{\partial v}{\partial x} \right)^2 + \frac{\partial g}{\partial u} \frac{\partial^2 u}{\partial x^2} + \frac{\partial g}{\partial v} \frac{\partial^2 v}{\partial x^2}$$

análogamente se tiene:

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 g}{\partial u^2} \left(\frac{\partial u}{\partial y} \right)^2 + 2 \left(\frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) \frac{\partial^2 g}{\partial u \partial v} + \frac{\partial^2 g}{\partial v^2} \left(\frac{\partial v}{\partial y} \right)^2 + \frac{\partial g}{\partial u} \frac{\partial^2 u}{\partial y^2} + \frac{\partial g}{\partial v} \frac{\partial^2 v}{\partial y^2}$$

Por lo tanto

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= \frac{\partial^2 g}{\partial u^2} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] + 2 \frac{\partial^2 g}{\partial u \partial v} \left[\left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \right) + \left(\frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) \right] + \frac{\partial^2 g}{\partial v^2} \left[\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] \\ &\quad + \frac{\partial g}{\partial u} \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] + \frac{\partial g}{\partial v} \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right] = 0 \end{aligned}$$

donde

$$\left. \begin{array}{l} \frac{\partial u}{\partial x} = y = \sqrt{\frac{u}{v}} \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \\ \frac{\partial u}{\partial y} = x = \sqrt{uv} \Rightarrow \frac{\partial^2 u}{\partial y^2} = 0 \end{array} \right| \quad \begin{array}{l} \frac{\partial v}{\partial x} = \frac{1}{y} = \sqrt{\frac{u}{v}} \Rightarrow \frac{\partial^2 v}{\partial x^2} = 0 \\ \frac{\partial v}{\partial y} = \frac{-x}{y^2} = \frac{-v}{u} \sqrt{uv} \Rightarrow \frac{\partial^2 v}{\partial y^2} = \frac{2x}{y^3} = \frac{2\sqrt{uv}}{\left(\frac{u}{v}\right)^{3/2}} \end{array}$$