Control #1 2003-2

1)

a) $A=\{(x_1,\dots,x_n)\in\mathbb{R}^n / x_i\in\mathbb{Q}\}$

$Adh(A)=\mathbb{R}^n$ $\quad Int(A)=\emptyset$ $\quad Fr(A)=\mathbb{R}^n$

b) Supongamos que A es abierto

Sea $x\in Int(Fr(A))$

$\Rightarrow \exists r>0 \quad B(x,r)\subset Fr(A) = Adh(A)/ Int(A)$

$= Adh(A) / A$

$\Rightarrow x\in Adh(A)$

$\Rightarrow B(x,r)\cap A\neq\emptyset \longrightarrow\!\!\leftarrow \Rightarrow Fr(A)=\emptyset$

Supongamos que A es cerrado

$\Rightarrow \mathbb{R}^n\setminus A$ es abierto

$\Rightarrow Fr(\mathbb{R}^n/A)=\emptyset$ $\quad$ pero $Fr(\mathbb{R}^n/A)=Fr(A)$

$\Rightarrow Fr(A)=\emptyset$.

En general no es cierto. Contraejemplo. $A=\{(x_1,x_2,\dots,x_n)\in\mathbb{R}^n / x_i\in\mathbb{Q}\}$

c) $A=\{(x,y) / x^2+y^2<1 ; x^2+y^2>1/4\}\cup\{(x,0) / 1\leq x<2\}$

$Int(A)=\{(x,y)/ x^2+y^2<1; x^2+y^2>1/4\}$
$Adh(A)=\{(x,y)/ x^2+y^2\leq 1; x^2+y^2\geq 1/4\}\cup\{(x,0)/ 1\leq x\leq 2\}$
$Fr(A)=\{(x,y)/ x^2+y^2=1 ; x^2+y^2\geq\frac{1}{4}\}\cup\{(x,0)/ 1\leq x\leq 2\}$

Como $A\neq Int(A)$ y $A\neq Adh(A) \Rightarrow$ A no es abierto
A no es cerrado

Control #1 2003-2

2]

a) $$f(x,y) = \begin{cases} \dfrac{1+x-\cos(x^2+y^2)-\operatorname{arctg} x}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

f es continua en $(0,0)$.

PDQ $\forall \varepsilon>0 \ \exists \delta>0 \ / \ \|\vec{x}\|<\delta \Rightarrow |f(x,y)-0|<\varepsilon$

$$\left|\frac{1+x-\cos(x^2+y^2)-\operatorname{arctg}(x)}{x^2+y^2}\right| \leq \left|\frac{1-\cos(x^2+y^2)}{x^2+y^2} + \frac{x-\operatorname{arctg}(x)}{x^2+y^2}\right|$$

$$= \left|\frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}(x^2+y^2) + \frac{x-\operatorname{arctg}(x)}{x^3}\,\frac{x^3}{x^2+y^2}\right|$$

Pero $\lim\limits_{z\to 0} \dfrac{1-\cos z}{z^2} = \dfrac{1}{2}$ $\quad \lim\limits_{z\to 0} \dfrac{z-\operatorname{arctg}(z)}{z^3} = \dfrac{1}{3}$

$\Rightarrow \left|\dfrac{1-\cos(x^2+y^2)}{x^2+y^2}\right| \leq 1$ para $\|\vec{x}\|<\sigma$ $\quad \left|\dfrac{x-\operatorname{arctg} x}{x^3}\right| \leq 1$ para $|x|<\sigma$

Por lo tanto si $\|\vec{x}\|<\sigma$

$$\left|\frac{1+x-\cos(x^2+y^2)-\operatorname{arctg} x}{x^2+y^2}\right| \leq \left|(x^2+y^2) + \frac{x^3}{x^2+y^2}\right| \leq \left|(x^2+y^2) + \frac{(x^2+y^2)^{3/2}}{x^2+y^2}\right|$$

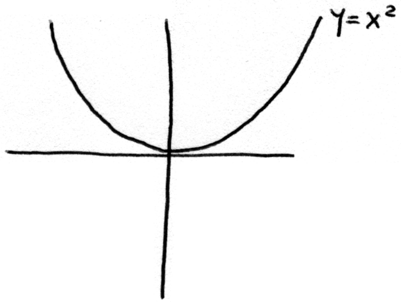
$$\leq \sigma^2 + \sigma$$

Basta tomar $\delta = \min\{1, \sigma, \varepsilon/2\}$

Control #1 2003-2

2| b)

$$f(x,y)=\begin{cases} 0 & y\le 0 \ \vee \ y\geq x^2 \\ 1 & 0<y<x^2 \end{cases}$$



f no es continua en $(0,0)$

Basta tomar

$y=x^2 \qquad \lim\limits_{x\to 0} f(x,x^2)=0$

$y=\frac{x^2}{2} \qquad \lim\limits_{x\to 0} f(x,\frac{x^2}{2})=1$

c)

$$f(x,y)=\begin{cases} \dfrac{y^2(x^3+y^2)+x^4}{x^4+y^4} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

$$f(x,y)=\frac{y^2x^3}{x^4+y^4}+1$$

f no es continua en $(0,0)$

$x=r\cos\theta \quad y=r\,\mathrm{sen}\,\theta$

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{r\to 0}\left[1+\frac{r^2\mathrm{sen}^2\theta\cdot r^3\cos^3\theta}{r^4(\mathrm{sen}^4\theta+\cos^4\theta)}\right]=\lim_{r\to 0} 1+r\cdot\frac{\mathrm{sen}^2\theta\cos^3\theta}{\mathrm{sen}^4\theta+\cos^4\theta}=1$$

Control #1 2003-2

3| a)
$$f(x,y)=\begin{cases}\dfrac{x^4+\operatorname{sen}(y^4)}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

$$\frac{\partial f}{\partial x}=\frac{2x^5+4x^3y^2-2x\operatorname{sen}y^4}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}=\frac{4y^3(x^2+y^2)\cos y^4-2x^4y-2y\operatorname{sen}y^4}{(x^2+y^2)^2}$$

Claramente $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ son continuas en $\mathbb{R}^2/\{(0,0)\}$ $\Rightarrow$ f es diferenciable en $\mathbb{R}^2/\{(0,0)\}$

$\Rightarrow$ f es continua en $\mathbb{R}^2/\{(0,0)\}$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t}=\lim_{t\to 0}\frac{\frac{t^4+\operatorname{sen}0^2}{t^2+0^2}}{t}=0$$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{t\to 0}\frac{f(0,t)-f(0,0)}{t}=\lim_{t\to 0}\frac{\operatorname{sen}t^4}{t^3}=0$$

$$\left|\frac{\partial f}{\partial x}(x,y)-\frac{\partial f}{\partial x}(0,0)\right|<\frac{2|x|^5+4|x|^3|y|^2+2|x||y|^4\left|\frac{\operatorname{sen}y^4}{y^4}\right|}{(x^2+y^2)^2}$$

Pero
$|x|\leq\sqrt{x^2+y^2}$
$|y|\leq\sqrt{x^2+y^2}$

$$<\frac{2(x^2+y^2)^{5/2}+4(x^2+y^2)^{5/2}+2(x^2+y^2)^{5/2}}{(x^2+y^2)^2}=8\sqrt{x^2+y^2}<8\delta=\varepsilon$$

$$\left|\frac{\partial f}{\partial y}(x,y)-\frac{\partial f}{\partial y}(0,0)\right|<\frac{4|y|^3(x^2+y^2)|\cos y^4|-2|x|^4|y|-2|y|\cdot|y|^4\left|\frac{\operatorname{sen}y^4}{y^4}\right|}{(x^2+y^2)^2}$$

$$<5\,\frac{(x^2+y^2)^{3/2}(x^2+y^2)-2(x^2+y^2)^{4/2}(x^2+y^2)^{1/2}-2(x^2+y^2)^{5/2}}{(x^2+y^2)^2}$$

$$<5\,\frac{(x^2+y^2)^{5/2}-2(x^2+y^2)^{5/2}-2(x^2+y^2)^{5/2}}{(x^2+y^2)^2}=\sqrt{x^2+y^2}<\delta=\varepsilon$$

$\Rightarrow \frac{\partial f}{\partial x}(0,0)\ \frac{\partial f}{\partial y}(0,0)$ son continuas en $(0,0)$ $\Rightarrow$ f diferenciable en $(0,0)$ $\Rightarrow$ f es continua en $(0,0)$

Control # 2003 2

3)

b) $\frac{\partial f}{\partial x_k}$ $f_k'(x_k$ $\forall$

PDQ

$$\lim_{y\to 0} \frac{f(x+y) - f(x) \quad \nabla f \cdot y}{y\|} \quad 0$$

En efecto

$$\lim_{y\to 0} \frac{\sum_{i=1} \left(f_i(x_i+y_i) - f_i(x_i) \right) - \sum_{i=1} f_i'(x_i)\cdot y_i}{\| y\|}$$

$\|y\| = |y_1| \quad |y_2|$

$$\lim_{y\to 0} \sum_{i=1}^{n} \frac{\left(f_i(x_i \quad y_i) \quad f_i(x_i) \quad f_i'(x_i) y_i \right)}{y_i} \quad 0$$

Ya que f_i diferenciable en Ja I