Control #1

1|

a) $N(p) = \int_0^1 |p'(t)|\,dt + |p(0)|$ con $p(t) = \sum_{k=0}^{n} a_k \cancel{p} t^k$

$N(p)$ es norma.

i) $0 \leq N(p) < +\infty$ directo

ii) $N(p) = 0 \iff p(t) = 0$

$\Leftarrow$| trivial.

$\Rightarrow$| $N(p) = 0 \Rightarrow \int_0^1 |p'(t)|\,dt + |p(0)| = 0$

$\Rightarrow \int_0^1 |p'(t)|\,dt = -|p(0)| \leq 0$

$\Rightarrow \int_0^1 |p'(t)|\,dt = 0 \Rightarrow p'(t) = 0 \quad \forall t \in [0,1]$

$\Rightarrow p(t) = cte \cdot$ ~~pero~~ $\Rightarrow p(t) = p(0)$

pero $|p(0)| = 0 \Rightarrow p(0) = 0 \Rightarrow p(t) = 0 \ \forall t.$ $\blacksquare$

iii) $N(\lambda p) = |\lambda| N(p)$

$N(\lambda p(t)) = \int_0^1 |\lambda p'(t)|\,dt + |\lambda p(0)| = |\lambda| \left\{ \int_0^1 |p'(t)|\,dt + |p(0)| \right\} = |\lambda| N(p(t))$ $\blacksquare$

iv) $N(p(t) + q(t)) \leq N(p(t)) + N(q(t))$

$|p'(t) + q'(t)| \leq |p'(t)| + |q'(t)| \quad / \int_0^1$ y $|p(0) + q(0)| \leq |p(0)| + |q(0)|$

$\Rightarrow \int_0^1 |p'(t) + q'(t)|\,dt + |p(0) + q(0)| \leq \int_0^1 |p'(t)|\,dt + \int_0^1 |q'(t)|\,dt + |p(0)| + |q(0)|$

$\leq N(p(t)) + N(q(t))$ $\blacksquare$

1| b) PDQ: $\exists L \in \mathbb{R} \,/\, \forall p \in E \quad N(p) \leq L \cdot \max_{i=0,\dots,n} |a_i|$

$$\text{Sea } p(t) = \sum_{k=0}^{n} a_k t^k \Rightarrow p'(t) = \sum_{k=1}^{n} k a_k t^{k-1}$$

$$N(p(t)) = \int_0^1 |p'(t)|\,dt + |p(0)|$$

$$= \int_0^1 \left| \sum_{k=1}^{n} k a_k t^{k-1} \right| dt + |a_0|$$

$$\leq \int_0^1 \sum_{k=1}^{n} k |a_k| t^{k-1} dt + |a_0|$$

$$\leq \sum_{k=1}^{n} k |a_k| \int_0^1 t^{k-1} dt + |a_0|$$

$$\leq \sum_{k=1}^{n} k |a_k| \left. \frac{t^k}{k} \right|_0^1 + |a_0| \leq \sum_{k=1}^{n} |a_k| + |a_0|.$$

$$\text{Sea } \quad a_{MAX} = \max_{i=0,\dots,n} |a_i|$$

$$\leq n \cdot a_{MAX} + a_{MAX} = \underbrace{(n+1)}_{L} a_{MAX}$$

$$N(p(t)) \leq L \cdot a_{MAX}. \quad \blacksquare$$

c) $M = \{ \lambda p(t) \,/\, \lambda \in \mathbb{R} \}$ $\quad p(t) = 1 + t + t^2 + \dots + t^n$

Calculemos $M \cap B_N(0,1)$

$$M \cap B_N(0,1) = \{ \lambda p(t) \in M \,/\, N(\lambda p(t)) \leq 1 \}$$

$$N(\lambda p) < 1 \Rightarrow |\lambda| N(p) < 1 \Rightarrow N(p) < \frac{1}{|\lambda|} \quad \text{con } |\lambda| \neq 0$$

Si $\lambda = 0 \Rightarrow \lambda p = 0 \Rightarrow N(\lambda p) = 0 \Rightarrow p = 0 \in M \cap B_N(0,1)$

$$\text{Pero } N(p) = \int_0^1 |\underbrace{1 + 2t + 3t^2 + \dots + nt^{n-1}}_{>0}|\,dt + |1| = \int_0^1 (1 + 2t + 3t^2 + \dots + nt^{n-1})\,dt + 1$$

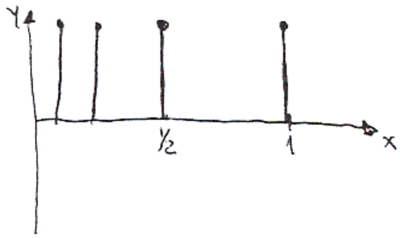
$$= 1t + 2\frac{t^2}{2} + 3\frac{t^3}{3} + \dots + n\frac{t^n}{n} \Big|_0^1 + 1 = \underbrace{1 + 1 + \dots + 1}_{n \text{ veces}} + 1 = (n+1)$$

$$\Rightarrow \quad n+1 < \frac{1}{|\lambda|} \Rightarrow |\lambda| < \frac{1}{n+1} \Rightarrow \lambda \in \left(-\frac{1}{n+1}, \frac{1}{n+1} \right) \Rightarrow M \cap B_N(0,1) = \{ \lambda p \,/\, \lambda \in (-\tfrac{1}{n+1}, \tfrac{1}{n+1}) \}$$

Control 1

2| a)

i) $A = \{ (x,y) \in \mathbb{R}^2 \,/\, x = \frac{1}{n} \quad n \in \mathbb{N} \,,\quad 0 \leq y \leq 1 \}$



$\text{Int}(A) = \emptyset \qquad \text{Adh}(A) = A \cup \{ (0,y) \;\; 0 \leq y \leq 1 \}$

ii) $B = \{ (x,y,z) \,/\, x+y+z = 1 \;;\; x^2+y^2+z^2 \leq 1 \}$

$\text{Adh}(B) = B \qquad \text{Int}(B) = \emptyset$

iii) $C = \{ (x,y) \in \mathbb{R}^2 \,/\, x^2 - y^2 < 1 \}$

$\text{Int}(C) = C \qquad \text{Adh}(C) = \{ (x,y) \,/\, x^2 - y^2 \leq 1 \}$

b) $\| A \|_\infty = \max\limits_{1 \leq i \leq 2} \{ \sum\limits_{j=1}^{2} |a_{ij}| \} \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$\Rightarrow \| A \|_\infty = \max \{ |a_{11}| + |a_{12}| \;;\; |a_{21}| + |a_{22}| \}$

i) $\| A \|_\infty$ es norma

I) $0 \leq \| A \|_\infty < +\infty$ directo.

II) $\| A \|_\infty = 0 \iff A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Leftarrow |$ directo

$\Rightarrow | \;\; \| A \|_\infty = 0 = \max \{ |a_{11}| + |a_{12}| \;;\; |a_{21}| + |a_{22}| \}$

Si $|a_{11}| + |a_{12}| = 0 \Rightarrow |a_{11}| = |a_{12}| = 0 \Rightarrow |a_{21}| = |a_{22}| = 0$

Si $|a_{21}| + |a_{22}| = 0 \Rightarrow |a_{21}| = |a_{22}| = 0 \Rightarrow |a_{11}| = |a_{12}| = 0$

Control 1

2| b) i)

III) $\|\lambda A\|_\infty = \max\{ |\lambda a_{11}| + |\lambda a_{12}| ; |\lambda a_{21}| + |\lambda a_{22}| \}$

$= \max\{ |\lambda| \{|a_{11}| + |a_{12}|\} ; |\lambda| \{|a_{21}| + |a_{22}|\} \}$

$= |\lambda| \{ |a_{11}| + |a_{12}| ; |a_{21}| + |a_{22}| \} = |\lambda| \|A\|_\infty$

IV) $\|A+B\|_\infty = \max\{ |a_{11}+b_{11}| + |a_{12}+b_{12}| ; |a_{21}+b_{21}| + |a_{22}+b_{22}| \}$

$\leq \max\{ |a_{11}| + |a_{12}| + |b_{11}| + |b_{12}| ; |a_{21}| + |a_{22}| + |b_{21}| + |b_{22}| \}$

$\leq \max\{ |a_{11}| + |a_{12}| ; |a_{21}| + |a_{22}| \} + \max\{ |b_{11}| + |b_{12}| ; |b_{21}| + |b_{22}| \}$

$\leq \|A\|_\infty + \|B\|_\infty$

$\therefore$ $\|A\|_\infty$ es norma en $M_{2\times 2}(\mathbb{R})$

ii) $A_n = \begin{bmatrix} (-1)^n & 1/n \\ 0 & 1 \end{bmatrix} \Rightarrow A_{2n} = \begin{bmatrix} 1 & \frac{1}{2n} \\ 0 & 1 \end{bmatrix} \longrightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

En efecto

$\| A_{2n} - A \|_\infty = \left\| \begin{bmatrix} 0 & 1/2n \\ 0 & 0 \end{bmatrix} \right\|_\infty = \frac{1}{2n} < \varepsilon$

Pero

$A_{2n+1} = \begin{bmatrix} -1 & \frac{1}{2n+1} \\ 0 & 1 \end{bmatrix} \longrightarrow A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$\| A_{2n+1} - A \|_\infty = \left\| \begin{bmatrix} 0 & \frac{1}{2n+1} \\ 0 & 0 \end{bmatrix} \right\| = \frac{1}{2n+1} < \varepsilon$

Si A_n converge entonces toda subsucesión converge al mismo límite

$\therefore$ A_n no es convergente.

Control #1

3)

a) $A \subset \mathbb{R}^n$ abierto ó cerrado $\Rightarrow Int(Fr(A)) = \emptyset$

Dem

$Fr(A) = \{ x \in \mathbb{R}^n / \forall \varepsilon > 0 \quad B(x,\varepsilon) \cap A \neq \emptyset \wedge B(x,\varepsilon) \cap A^c \neq \emptyset \}$

Es claro que $\forall \varepsilon > 0 \quad B(x,\varepsilon) \cap A \neq \emptyset$

pero $B(x,\varepsilon) \cap A^c \neq \emptyset \Rightarrow B(x,\varepsilon) \not\subset A \quad \forall \varepsilon > 0$

$\therefore Int(Fr(A)) = \emptyset$

No es válido para cualquier conjunto: Sea $A = \mathbb{Q}$ (racionales.)

b) $\rho(x,y)$ $\sigma(x,y)$ métricas.

1- $\beta(x,y) = \rho(x,y) + \sigma(x,y)$ es métrica

I- $0 \leq \beta(x,y) < +\infty$ directo

II- $\beta(x,y) = \rho(x,y) + \sigma(x,y) = \rho(y,x) + \sigma(y,x) = \beta(y,x)$ ✓

III $\beta(x,y) = 0 \Leftrightarrow x = y$

$\Leftarrow)$ si $x = y \Rightarrow \beta(x,y) = \beta(x,x) = \rho(x,x) + \sigma(x,x) = 0$

$\Rightarrow)$ $\beta(x,y) = 0 \Rightarrow \rho(x,y) + \sigma(x,y) = 0 \Rightarrow \rho(x,y) = -\sigma(x,y) \geq 0$

$\Rightarrow \rho(x,y) = \sigma(x,y) = 0 \Rightarrow x = y$

IV $\beta(x,y) \leq \beta(x,z) + \beta(z,y)$

$\beta(x,y) = \rho(x,y) + \sigma(x,y) \leq \rho(x,z) + \rho(z,y) + \sigma(x,z) + \sigma(y,z) = \beta(x,z) + \beta(z,y)$

2- Si $\forall x,y \in X \quad \rho(x,y) \leq \sigma(x,y) \Rightarrow S \subset X$ es cerrado c/r σ si lo es c/r a ρ.

Sea $x_n \to x$ con la métrica $\sigma \Rightarrow \lim \sigma(x_n, x) = 0$

PDQ. $x \in S$

$0 \leq \lim \rho(x_n,x) \leq \lim \sigma(x_n,x) \Rightarrow \lim \rho(x_n,x) = 0$ pero S es cerrado c/r a $\rho \Rightarrow x \in S \quad \therefore S$ es cerrado c/r. a ρ.

c) $$I \lim_{\vec{x} \to \vec{0}} \frac{(xy)^2}{(xy)^2 + (x-y)^2}$$

Sea $x=0$ $y=y$ $\Rightarrow$ $I = \lim_{y \to 0} \frac{0}{0+y^2} = 0$

$x=y$ $\Rightarrow$ $I = \lim_{x \to 0} \frac{x^4}{x^4 - 0} = 1$

$\Rightarrow$ f no es continua en $(0,0)$