# LINEAR FUNCTIONAL ANALYSIS


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# Chapter 9

## SPECTRUM OF A LINEAR OPERATOR

### 9.1. Introduction

Suppose that $V$ is a Banach space over $\mathbb{F}$, and that $T \in B(V)$ is a continuous linear operator. We are interested in the set of values $\lambda \in \mathbb{F}$ such that the linear operator $\lambda I - T$ is not invertible, where $I \in B(V)$ denotes the identity linear operator on $V$.

Definition. Suppose that $V$ is a Banach space over $\mathbb{F}$, and that $T \in B(V)$. The set

$$\sigma(T) = \{\lambda \in \mathbb{F} : \lambda I - T \text{ is not invertible}\}$$

is called the spectrum of the linear operator $T$.

Example 9.1.1. Suppose that $V$ is a finite dimensional normed vector space. Then we recall from linear algebra that every continuous linear operator $T$ on $V$ can be described in terms of a square matrix $A$, and that the linear operator $\lambda I - T$ is invertible precisely when the matrix $\lambda I - A$ is invertible. It follows that the spectrum $\sigma(T)$ consists precisely of all the eigenvalues of the linear operator $T$, and these are precisely the eigenvalues of the matrix $A$.

Example 9.1.2. Suppose that $I$ is the identity linear operator on a Banach space $V$. Then clearly $\sigma(I) = \{1\}$, since $\lambda I - I = (\lambda - 1)I$ is invertible unless $\lambda - 1 = 0$.

**THEOREM 9A.** *Suppose that $V$ is a Banach space over $\mathbb{F}$, and that $T \in B(V)$. Then every eigenvalue of $T$ belongs to $\sigma(T)$.*

Proof. Note that $\lambda \in \mathbb{F}$ is an eigenvalue of $T$ if there exists a non-zero vector $\mathbf{x} \in V$ such that $T(\mathbf{x}) = \lambda\mathbf{x} = \lambda I(\mathbf{x})$, so that $(\lambda I - T)(\mathbf{x}) = \mathbf{0}$, whence $\lambda I - T$ has non-trivial kernel and cannot therefore be invertible. ♣

To highlight the difference between finite dimensional and infinite dimensional Banach spaces, we consider the following example.

Example 9.1.3. Consider the continuous linear operator $T : \ell^2 \to \ell^2$ on the Hilbert space $\ell^2$ of all square summable infinite sequences of complex numbers, given by

$$T(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots) \qquad \text{for every } (x_1, x_2, x_3, \ldots) \in \ell^2,$$

as discussed in Example 7.1.6. Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue of $T$. Then there exists a non-zero eigenvector $(x_1, x_2, x_3, \ldots) \in \ell^2$ such that

$$(0, x_1, x_2, x_3, \ldots) = (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4 \ldots),$$

so that $\lambda x_1 = 0$ and $\lambda x_i = x_{i-1}$ for every $i > 1$. If $\lambda = 0$, then the second condition implies that $x_1 = x_2 = x_3 = \ldots = 0$, a contradiction. If $\lambda \neq 0$, then the first condition implies $x_1 = 0$, and the second condition implies $x_2 = x_3 = x_4 = \ldots = 0$, a contradiction again. It follows that the linear operator $T$ has no eigenvalues.

The following two results go some way towards helping us find the spectrum of a continuous linear operator.

**THEOREM 9B.** *Suppose that $V$ is a Banach space over $\mathbb{F}$, and that $T \in B(V)$. Suppose further that $\lambda \in \mathbb{F}$ satisfies $|\lambda| > \|T\|$. Then $\lambda \notin \sigma(T)$. In other words, we have $\sigma(T) \subseteq \{\lambda \in \mathbb{F} : |\lambda| \le \|T\|\}$, so that the spectrum $\sigma(T)$ is contained in the closed disc of radius $\|T\|$ and centre $0$ in $\mathbb{F}$.*

Proof. The condition $|\lambda| > \|T\|$ implies that $\|\lambda^{-1}T\| < 1$. It follows from Theorem 7G that $I - \lambda^{-1}T$ is invertible, and so $\lambda I - T$ is invertible, whence $\lambda \notin \sigma(T)$. ♣

**THEOREM 9C.** *Suppose that $V$ is a Banach space over $\mathbb{F}$, and that $T \in B(V)$. Then the spectrum $\sigma(T)$ is closed in $\mathbb{F}$.*

Proof. Consider the function $f : \mathbb{F} \to B(V)$, given by $f(\lambda) = \lambda I - T$ for every $\lambda \in \mathbb{F}$. Then

$$\|f(\mu) - f(\lambda)\| = \|(\mu I - T) - (\lambda I - T)\| = |\mu - \lambda| \qquad \text{for every } \mu, \lambda \in \mathbb{F},$$

and so $f : \mathbb{F} \to B(V)$ is continuous. Next, recall Theorem 7E, that the set of all invertible linear operators in $B(V)$ is open, and so the set $\mathcal{C}$ of all non-invertible linear operators in $B(V)$ is closed. It now follows from Theorem 1J that $\sigma(T) = f^{-1}(\mathcal{C})$ is closed. ♣

## 9.2. Compact Operators

We have shown that for any continuous linear operator $T$ on a Banach space over $\mathbb{F}$, the spectrum $\sigma(T)$ is contained in the closed disc of radius $\|T\|$ and centre 0 in $\mathbb{F}$. We shall now show that there are instances where there are points of the spectrum $\sigma(T)$ on the boundary of this closed disc.

Definition. Suppose that $V$ is a normed vector space over $\mathbb{F}$. A linear operator $T : V \to V$ is said to be compact if, for every bounded sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in $V$, the sequence $(T(\mathbf{x}_n))_{n\in\mathbb{N}}$ has a convergent subsequence.

Remark. Note that a compact linear operator $T$ on $V$ is necessarily continuous. For otherwise, there exists a bounded sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in $V$ such that $\|T(\mathbf{x}_n)\| \to \infty$ as $n \to \infty$, and so $(T(\mathbf{x}_n))_{n\in\mathbb{N}}$ clearly cannot have a convergent subsequence.

The following example is a clear indication of the sort of problems we face in infinite dimensional normed vector spaces.

EXAMPLE 9.2.1. Suppose that $V$ is an infinite dimensional normed vector space over $\mathbb{F}$. Then the identity linear operator $I : V \to V$ is not compact. To see this, recall from Example 3.5.4 that there exists an infinite sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ of unit vectors in $V$ which does not have any convergent subsequence, so that the sequence $(I(\mathbf{x}_n))_{n\in\mathbb{N}}$ does not have any convergent subsequence.

**THEOREM 9D.** *Suppose that $T$ is a compact Hermitian operator on a Hilbert space $V$ over $\mathbb{F}$. Then at least one of $\pm\|T\|$ is an eigenvelue of $T$, and so belongs to $\sigma(T)$.*

PROOF. The proof is trivial if $T$ is the zero linear operator on $V$, so we shall assume that $T$ is not the zero linear operator on $V$. Recall Theorem 8B, that

$$\|T\| = \sup_{\mathbf{x}\in V,\ \|\mathbf{x}\|=1} |\langle T(\mathbf{x}), \mathbf{x}\rangle|.$$

Hence there exists a sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ such that $\|\mathbf{x}_n\| = 1$ for every $n \in \mathbb{N}$ and $|\langle T(\mathbf{x}_n), \mathbf{x}_n\rangle| \to \|T\|$ as $n \to \infty$. Observe next that since $T$ is Hermitian, we have

$$\langle T(\mathbf{x}_n), \mathbf{x}_n\rangle = \langle \mathbf{x}_n, T^*(\mathbf{x}_n)\rangle = \langle \mathbf{x}_n, T(\mathbf{x}_n)\rangle = \overline{\langle T(\mathbf{x}_n), \mathbf{x}_n\rangle},$$

so that $\langle T(\mathbf{x}_n), \mathbf{x}_n\rangle$ is real. Replacing $(\mathbf{x}_n)_{n\in\mathbb{N}}$ by a subsequence if necessary, we may therefore assume that $\langle T(\mathbf{x}_n), \mathbf{x}_n\rangle \to \lambda$ as $n \to \infty$, where $\lambda = \|T\|$ or $\lambda = -\|T\|$. Then

$$\begin{aligned} \|T(\mathbf{x}_n) - \lambda\mathbf{x}_n\|^2 &= \langle T(\mathbf{x}_n) - \lambda\mathbf{x}_n, T(\mathbf{x}_n) - \lambda\mathbf{x}_n\rangle = \|T(\mathbf{x}_n)\|^2 - 2\lambda\langle T(\mathbf{x}_n), \mathbf{x}_n\rangle + \lambda^2\|\mathbf{x}_n\|^2 \\ &\le \|T\|^2\|\mathbf{x}_n\|^2 - 2\lambda\langle T(\mathbf{x}_n), \mathbf{x}_n\rangle + \lambda^2\|\mathbf{x}_n\|^2 = 2\lambda^2 - 2\lambda\langle T(\mathbf{x}_n), \mathbf{x}_n\rangle, \end{aligned}$$

so that

$$0 \le \|T(\mathbf{x}_n) - \lambda\mathbf{x}_n\|^2 \le 2\lambda^2 - 2\lambda\langle T(\mathbf{x}_n), \mathbf{x}_n\rangle \to 0 \qquad \text{as } n \to \infty.$$

It follows that $T(\mathbf{x}_n) - \lambda\mathbf{x}_n \to 0$ as $n \to \infty$. We next make use of the compactness of the linear operator $T$ to conclude that there exists a subsequence $(\mathbf{x}_{n_p})_{p\in\mathbb{N}}$ of $(\mathbf{x}_n)_{n\in\mathbb{N}}$ such that $(T(\mathbf{x}_{n_p}))_{p\in\mathbb{N}}$ is a convergent sequence. Let $T(\mathbf{x}_{n_p}) \to \mathbf{y}$ as $p \to \infty$. Then $\lambda\mathbf{x}_{n_p} \to \mathbf{y}$ as $p \to \infty$, and so $\lambda T(\mathbf{x}_{n_p}) \to T(\mathbf{y})$ as $p \to \infty$, giving $T(\mathbf{y}) = \lambda\mathbf{y}$. Note finally that

$$\|\mathbf{y}\| = \lim_{p\to\infty} \|\lambda\mathbf{x}_{n_p}\| = |\lambda| = \|T\| \ne 0,$$

so that $\lambda$ is an eigenvalue of $T$. ♣

## PROBLEMS FOR CHAPTER 9

1. Show that $\sigma(T) = \overline{\{c_n : n \in \mathbb{N}\}}$ in Problem 3 in Chapter 7 if the sequence $(c_n)_{n\in\mathbb{N}}$ is bounded.

2. a) Show that every $\lambda \in \mathbb{C}$ satisfying $|\lambda| < 1$ is an eigenvalue of the linear operator $S : \ell^2 \to \ell^2$ in Example 7.3.1.
   b) Find the spectrum $\sigma(S)$.

3. Suppose that $V$ is a Hilbert space over $\mathbb{F}$, and that $T \in B(V)$. Show that for the adjoint operator $T^* : V \to V$, we have $\sigma(T^*) = \{\overline{\lambda} : \lambda \in \sigma(T)\}$.

4. Find the spectrum $\sigma(T)$ of the linear operator $T : \ell^2 \to \ell^2$ in Example 7.3.1.
   [HINT: Refer to Problem 6(a) in Chapter 8.]

5. Suppose that $V$ is a Banach spoace over $\mathbb{F}$, and that $T \in B(V)$. By using the identity

$$(\lambda I - T)(\lambda I + T) = \lambda^2 I - T^2,$$

show that $\sigma(T^2) = \{\lambda^2 : \lambda \in \sigma(T)\}$.

6. Consider the bounded linear operator $T : \ell^2 \to \ell^2$, defined by

$$T(x_1, x_2, x_3, x_4, \ldots) = (x_1, -x_2, x_3, -x_4, \ldots) \qquad \text{for every } (x_1, x_2, x_3, x_4, \ldots) \in \ell^2.$$

a) Show that $\pm 1$ are eigenvalues of $T$.
b) Find $T^2$ and hence show that $\sigma(T) = \{\pm 1\}$.

7. Consider the bounded linear operator $T : \ell^2 \to \ell^2$, defined by

$$T(x_1, x_2, x_3, x_4, \ldots) = (0, x_1, 0, x_3, \ldots) \qquad \text{for every } (x_1, x_2, x_3, x_4, \ldots) \in \ell^2.$$

a) Show that 0 is an eigenvalue of $T$.
b) Find $T^2$ and hence show that $\sigma(T) = \{0\}$.

8. Suppose that $V$ is a normed vector space over $\mathbb{F}$, and that $T, S \in B(V)$.
a) Show that $ST$ is compact if $T$ is compact.
b) Show that $ST$ is compact if $S$ is compact.
c) Show that if $T$ is compact, then $T$ is not invertible.

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