LINEAR FUNCTIONAL ANALYSIS

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Chapter 8

LINEAR TRANSFORMATIONS ON HILBERT SPACES

8.1. Adjoint Transformations

We begin with a result which is a consequence of the Riesz-Fréchet theorem first studied in Section 6.3.

THEOREM 8A. *Suppose that V and W are Hilbert spaces over $\mathbb{F}$. For every linear transformation $T \in B(V,W)$, there exists a unique linear transformation $T^* \in B(W,V)$ such that*

$$\langle T(\mathbf{x}), \mathbf{y}\rangle = \langle \mathbf{x}, T^*(\mathbf{y})\rangle \qquad \textit{for every } \mathbf{x} \in V \textit{ and } \mathbf{y} \in W.$$

Remark. Note that in the above, $\langle T(\mathbf{x}), \mathbf{y}\rangle$ is an inner product in the Hilbert space W, while $\langle \mathbf{x}, T^*(\mathbf{y})\rangle$ is an inner product in the Hilbert space V.

Definition. Suppose that V and W are Hilbert spaces over $\mathbb{F}$. The unique linear transformation $T^* \in B(W,V)$ satisfying the conclusion of Theorem 8A is called the adjoint transformation of the linear transformation $T \in B(V,W)$.

Proof of Theorem 8A. Suppose that $\mathbf{y} \in W$ is fixed. It is easy to check that the mapping $S : V \to \mathbb{F}$, given for every $\mathbf{x} \in V$ by $S(\mathbf{x}) = \langle T(\mathbf{x}), \mathbf{y}\rangle$, is a linear functional on V. Furthermore, we have

$$|S(\mathbf{x})| = |\langle T(\mathbf{x}), \mathbf{y}\rangle| \le \|T(\mathbf{x})\|\|\mathbf{y}\| \le \|T\|\|\mathbf{x}\|\|\mathbf{y}\| = (\|T\|\|\mathbf{y}\|)\|\mathbf{x}\| \qquad \text{for every } \mathbf{x} \in V,$$

so that $S : V \to \mathbb{F}$ is a bounded, and continuous, linear functional on V. It follows from the Riesz-Fréchet theorem that there exists a unique $\mathbf{u} \in V$ such that $S(\mathbf{x}) = \langle \mathbf{x}, \mathbf{u}\rangle$ for every $\mathbf{x} \in V$. Write $\mathbf{u} = T^*(\mathbf{y})$. Then $T^* : W \to V$ is a mapping satisfying

$$\langle T(\mathbf{x}), \mathbf{y}\rangle = \langle \mathbf{x}, T^*(\mathbf{y})\rangle \qquad \text{for every } \mathbf{x} \in V \text{ and } \mathbf{y} \in W.$$

Next we show that $T^* \in B(W,V)$. Suppose first of all that $\mathbf{y},\mathbf{z} \in W$ and $c \in \mathbb{F}$. Then for every $\mathbf{x} \in V$, we have

$$\begin{aligned}\langle \mathbf{x}, T^*(\mathbf{y}+\mathbf{z})\rangle &= \langle T(\mathbf{x}), \mathbf{y}+\mathbf{z}\rangle = \langle T(\mathbf{x}),\mathbf{y}\rangle + \langle T(\mathbf{x}),\mathbf{z}\rangle \\ &= \langle \mathbf{x}, T^*(\mathbf{y})\rangle + \langle \mathbf{x}, T^*(\mathbf{z})\rangle = \langle \mathbf{x}, T^*(\mathbf{y}) + T^*(\mathbf{z})\rangle\end{aligned}$$

and

$$\langle \mathbf{x}, T^*(c\mathbf{y})\rangle = \langle T(\mathbf{x}), c\mathbf{y}\rangle = \overline{c}\langle T(\mathbf{x}),\mathbf{y}\rangle = \overline{c}\langle \mathbf{x}, T^*(\mathbf{y})\rangle = \langle \mathbf{x}, cT^*(\mathbf{y})\rangle,$$

so that $T^*(\mathbf{y}+\mathbf{z}) = T^*(\mathbf{y}) + T^*(\mathbf{z})$ and $T^*(c\mathbf{y}) = cT^*(\mathbf{y})$ by Theorem 4B. It follows that $T^* : W \to V$ is a linear transformation. Furthermore, for every $\mathbf{y} \in W$, we have

$$\|T^*(\mathbf{y})\|^2 = \langle T^*(\mathbf{y}), T^*(\mathbf{y})\rangle = \langle T(T^*(\mathbf{y})), \mathbf{y}\rangle \le \|T(T^*(\mathbf{y}))\|\|\mathbf{y}\| \le \|T\|\|T^*(\mathbf{y})\|\|\mathbf{y}\|.$$

Suppose that $\|T^*(\mathbf{y})\| > 0$. Then dividing the above by $\|T^*(\mathbf{y})\|$, we obtain $\|T^*(\mathbf{y})\| \le \|T\|\|\mathbf{y}\|$. Note that this last inequality is satisfied trivially if $\|T^*(\mathbf{y})\| = 0$. It follows that

$$\|T^*(\mathbf{y})\| \le \|T\|\|\mathbf{y}\| \qquad \text{for every } \mathbf{y} \in W,$$

and so $T^* : W \to V$ is bounded, whence $T^* \in B(W,V)$. Finally, suppose that $T_1, T_2 \in B(W,V)$ satisfy

$$\langle T(\mathbf{x}), \mathbf{y}\rangle = \langle \mathbf{x}, T_1(\mathbf{y})\rangle = \langle \mathbf{x}, T_2(\mathbf{y})\rangle \qquad \text{for every } \mathbf{x} \in V \text{ and } \mathbf{y} \in W.$$

Then it follows from Theorem 4B that $T_1(\mathbf{y}) = T_2(\mathbf{y})$ for every $\mathbf{y} \in W$, so that $T_1 = T_2$. The uniqueness of $T^* \in B(W,V)$ follows immediately. ♣

Example 8.1.1. Suppose that $a, b \in \mathbb{R}$ and $a < b$. Consider the vector space $L^2[a,b]$ of all complex valued Lebesgue measurable functions that are square integrable on $[a,b]$. We know that the norm

$$\|f\| = \left(\int_a^b |f(t)|^2 \,\mathrm{d}t\right)^{1/2},$$

given in Example 7.1.3, is in fact induced by the inner product

$$\langle f, g\rangle = \left(\int_a^b f(t)\overline{g(t)} \,\mathrm{d}t\right)^{1/2}.$$

Let $\phi \in C[a,b]$ be chosen and fixed, and consider the bounded linear operator $T : L^2[a,b] \to L^2[a,b]$, where for every $f \in L^2[a,b]$, the function $T(f) \in L^2[a,b]$ is defined by

$$(T(f))(t) = \phi(t)f(t) \qquad \text{for every } t \in [a,b],$$

as discussed in Example 7.1.3. It follows from Theorem 8A that the adjoint operator T^* satisfies

$$\langle T(f), g\rangle = \langle f, T^*(g)\rangle \qquad \text{for every } f, g \in L^2[a,b].$$

In other words, we must have

$$\int_a^b \phi(t)f(t)\overline{g(t)} \,\mathrm{d}t = \int_a^b f(t)\overline{(T^*(g))(t)} \,\mathrm{d}t \qquad \text{for every } f, g \in L^2[a,b].$$

Clearly

$$(T^*(g))(t) = \overline{\phi(t)}g(t) \qquad \text{for every } t \in [a,b]$$

would be sufficient. Hence by uniqueness, the adjoint operator $T^* : L^2[a,b] \to L^2[a,b]$ is given for every $g \in L^2[a,b]$ by this.

EXAMPLE 8.1.2. Suppose that $a, b, c, d \in \mathbb{R}$, with $a < b$ and $c < d$. Consider the vector spaces $L^2[a, b]$ and $L^2[c, d]$. We know that the respective norms

$$\|f\| = \left(\int_a^b |f(t)|^2 \, \mathrm{d}t \right)^{1/2} \qquad \text{and} \qquad \|h\| = \left(\int_c^d |h(s)|^2 \, \mathrm{d}s \right)^{1/2},$$

given in Example 7.1.4, are in fact induced by the respective inner products

$$\langle f, g \rangle = \left(\int_a^b f(t)\overline{g(t)} \, \mathrm{d}t \right)^{1/2} \qquad \text{and} \qquad \langle h, k \rangle = \left(\int_c^d h(s)\overline{k(s)} \, \mathrm{d}s \right)^{1/2}.$$

Let $\phi : [c, d] \times [a, b] \to \mathbb{C}$ be a fixed continuous function, and consider the bounded linear transformation $T : L^2[a, b] \to L^2[c, d]$, where for every $f \in L^2[a, b]$, the function $T(f) \in L^2[c, d]$ is defined by

$$(T(f))(s) = \int_a^b \phi(s, t) f(t) \, \mathrm{d}t \qquad \text{for every } s \in [c, d],$$

as discussed in Example 7.1.4. It follows from Theorem 8A that the adjoint operator T^* satisfies

$$\langle T(f), k \rangle = \langle f, T^*(k) \rangle \qquad \text{for every } f \in L^2[a, b] \text{ and } k \in L^2[c, d].$$

In other words, we must have

$$\int_c^d \left(\int_a^b \phi(s, t) f(t) \, \mathrm{d}t \right) \overline{k(s)} \, \mathrm{d}s = \int_a^b f(t)\overline{(T^*(k))(t)} \, \mathrm{d}t \qquad \text{for every } f \in L^2[a, b] \text{ and } k \in L^2[c, d].$$

By Fubini's theorem, clearly

$$(T^*(k))(t) = \int_c^d \overline{\phi(s, t)} k(s) \, \mathrm{d}s \qquad \text{for every } t \in [a, b]$$

would be sufficient. Hence by uniqueness, the adjoint transformation $T^* : L^2[c, d] \to L^2[a, b]$ is given for every $k \in L^2[c, d]$ by this.

8.2. Hermitian Operators

We conclude our discussion by studying a special type of adjoint operators.

DEFINITION. Suppose that V is a Hilbert space over $\mathbb{F}$. A linear operator $T \in L(V)$ is said to be self-adjoint or Hermitian if $T^* = T$.

EXAMPLE 8.2.1. Suppose that $a, b \in \mathbb{R}$ and $a < b$. Consider the Hilbert space $L^2[a, b]$ of all complex valued Lebesgue measurable functions that are square integrable on $[a, b]$, as discussed in Example 8.1.1. Let $\phi \in C[a, b]$ be chosen and fixed. For the bounded linear operator $T : L^2[a, b] \to L^2[a, b]$, where for every $f \in L^2[a, b]$, the function $T(f) \in L^2[a, b]$ is defined by $(T(f))(t) = \phi(t)f(t)$ for every $t \in [a, b]$, we have shown earlier that the adjoint operator $T^* : L^2[a, b] \to L^2[a, b]$ is given for every $g \in L^2[a, b]$ by $(T^*(g))(t) = \overline{\phi(t)}g(t)$ for every $t \in [a, b]$. Hence $T : L^2[a, b] \to L^2[a, b]$ is Hermitian if $\phi \in C[a, b]$ is real valued.

The following result gives a technique for finding the norm of a Hermitian operator.

THEOREM 8B. *Suppose that V is a Hilbert space over $\mathbb{F}$. Suppose further that $T \in B(V)$ is an Hermitian operator. Then*

$$\|T\| = \sup_{\mathbf{x}\in V,\ \|\mathbf{x}\|=1} |\langle T(\mathbf{x}), \mathbf{x}\rangle|.$$

PROOF. For every $\mathbf{x} \in V$ satisfying $\|\mathbf{x}\| = 1$, we have

$$|\langle T(\mathbf{x}), \mathbf{x}\rangle| \le \|T(\mathbf{x})\|\|\mathbf{x}\| \le \|T\|\|\mathbf{x}\|^2 = \|T\|,$$

so that

$$\|T\| \ge \sup_{\mathbf{x}\in V,\ \|\mathbf{x}\|=1} |\langle T(\mathbf{x}), \mathbf{x}\rangle|.$$

To prove the opposite inequality, let

$$M = \sup_{\mathbf{x}\in V,\ \|\mathbf{x}\|=1} |\langle T(\mathbf{x}), \mathbf{x}\rangle|.$$

For any non-zero vector $\mathbf{u} \in V$, the vector $\mathbf{u}/\|\mathbf{u}\|$ has norm 1. It follows easily from linearity that

$$|\langle T(\mathbf{u}), \mathbf{u}\rangle| \le M\|\mathbf{u}\|^2 \qquad \text{for every } \mathbf{u} \in V.$$

For every $\mathbf{x}, \mathbf{y} \in V$, noting that $T^* = T$, it is not difficult to check that

$$\begin{aligned}
\langle T(\mathbf{x}+\mathbf{y}), \mathbf{x}+\mathbf{y}\rangle &= \langle T(\mathbf{x}) + T(\mathbf{y}), \mathbf{x}+\mathbf{y}\rangle \\
&= \langle T(\mathbf{x}), \mathbf{x}\rangle + \langle T(\mathbf{x}), \mathbf{y}\rangle + \langle T(\mathbf{y}), \mathbf{x}\rangle + \langle T(\mathbf{y}), \mathbf{y}\rangle \\
&= \langle T(\mathbf{x}), \mathbf{x}\rangle + \langle T(\mathbf{x}), \mathbf{y}\rangle + \langle \mathbf{y}, T^*(\mathbf{x})\rangle + \langle T(\mathbf{y}), \mathbf{y}\rangle \\
&= \langle T(\mathbf{x}), \mathbf{x}\rangle + \langle T(\mathbf{x}), \mathbf{y}\rangle + \overline{\langle T(\mathbf{x}), \mathbf{y}\rangle} + \langle T(\mathbf{y}), \mathbf{y}\rangle \\
&= \langle T(\mathbf{x}), \mathbf{x}\rangle + 2\mathfrak{Re}\langle T(\mathbf{x}), \mathbf{y}\rangle + \langle T(\mathbf{y}), \mathbf{y}\rangle,
\end{aligned}$$

and similarly

$$\langle T(\mathbf{x}-\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle = \langle T(\mathbf{x}), \mathbf{x}\rangle - 2\mathfrak{Re}\langle T(\mathbf{x}), \mathbf{y}\rangle + \langle T(\mathbf{y}), \mathbf{y}\rangle,$$

and so

$$\begin{aligned}
4\mathfrak{Re}\langle T(\mathbf{x}), \mathbf{y}\rangle &= \langle T(\mathbf{x}+\mathbf{y}), \mathbf{x}+\mathbf{y}\rangle - \langle T(\mathbf{x}-\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle \\
&\le M(\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2) \\
&= 2M(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2),
\end{aligned}$$

the last step as a consequence of the Parallelogram law. We can replace $\mathbf{x}$ by $\lambda\mathbf{x}$, where $\lambda \in \mathbb{F}$ satisfies $|\lambda| = 1$ and $\mathfrak{Re}\langle T(\lambda\mathbf{x}), \mathbf{y}\rangle = |\langle T(\mathbf{x}), \mathbf{y}\rangle|$. Then

$$|\langle T(\mathbf{x}), \mathbf{y}\rangle| \le \tfrac{1}{2}M(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2) \qquad \text{for every } \mathbf{x}, \mathbf{y} \in V.$$

Suppose first of all that $T(\mathbf{x}) \ne \mathbf{0}$. Then taking

$$\mathbf{y} = \frac{\|\mathbf{x}\|}{\|T(\mathbf{x})\|} T(\mathbf{x}),$$

we have $\|\mathbf{y}\| = \|\mathbf{x}\|$ and

$$\frac{\|\mathbf{x}\|}{\|T(\mathbf{x})\|} |\langle T(\mathbf{x}), T(\mathbf{x})\rangle| \le M\|\mathbf{x}\|^2, \qquad \text{so that} \qquad \|T(\mathbf{x})\| \le M\|\mathbf{x}\|.$$

The last inequality holds trivially if $T(\mathbf{x}) = \mathbf{0}$, and therefore holds for every $\mathbf{x} \in V$. It follows that $\|T\| \le M$ as required. ♣

Problems for Chapter 8

1. Suppose that V and W are Hilbert spaces over $\mathbb{F}$, and that $T \in B(V,W)$.
   a) Show that $\langle \mathbf{y}, (T^*)^*(\mathbf{x})\rangle = \langle \mathbf{y}, T(\mathbf{x})\rangle$ for every $\mathbf{x} \in V$ and $\mathbf{y} \in W$, using condition (IP1) twice. Explain why this implies that $(T^*)^* = T$.
   b) Deduce that $\|T^*\| = \|T\|$. You may wish to study the proof of Theorem 8A for some useful information.
   c) Show that $\|T(\mathbf{x})\|^2 \le \|T^*T\|\|\mathbf{x}\|^2$ for every $\mathbf{x} \in V$. Explain why this implies the inequality $\|T\|^2 \le \|T^*T\|$.
   d) Deduce that $\|T^*T\| = \|T\|^2$.

2. Suppose that V, W and U are Hilbert spaces over $\mathbb{F}$, and that $T \in B(V,W)$ and $S \in B(W,U)$. Show that $(ST)^* = T^*S^*$.

3. Suppose that V and W are Hilbert spaces over $\mathbb{F}$. Show that for every $c, a \in \mathbb{F}$ and $T, S \in B(V,W)$, we have $(cT + aS)^* = \overline{c}T^* + \overline{a}S^*$.

4. Suppose that V and W are Hilbert spaces over $\mathbb{F}$. Show that the function $f : B(V,W) \to B(W,V)$, defined for every $T \in B(V,W)$ by $f(T) = T^*$, is continuous in $B(V,W)$.
   [Hint: Show that $\|f(T) - f(S)\| = \|T - S\|$ for every $T, S \in B(V,W)$.]

5. Suppose that V is a complex Hilbert space, and that $\mathbf{x}_1, \mathbf{x}_2 \in V$ are fixed. Consider the bounded linear operator $T : V \to V$, where $T(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x}_1\rangle \mathbf{x}_2$ for every $\mathbf{x} \in V$. Show that the adjoint operator $T^* : V \to V$ is given by $T^*(\mathbf{y}) = \langle \mathbf{y}, \mathbf{x}_2\rangle \mathbf{x}_1$ for every $\mathbf{y} \in V$.

6. Consider the vector space ℓ^2 of all square summable infinite sequences of complex numbers, with inner product
$$\langle \mathbf{x}, \mathbf{y}\rangle = \left(\sum_{i=1}^{\infty} x_i \overline{y_i}\right)^{1/2}.$$
   For each of the given bounded linear operators $T : \ell^2 \to \ell^2$, find the adjoint operator $T^* : \ell^2 \to \ell^2$:
   a) $T(\mathbf{x}) = (0, x_1, x_2, x_3, \ldots)$ for every $\mathbf{x} = (x_1, x_2, x_3, \ldots)$, as discussed in Example 7.1.6.
   b) $T(\mathbf{x}) = (0, 2x_1, x_2, 2x_3, x_4, \ldots)$ for every $\mathbf{x} = (x_1, x_2, x_3, x_4, \ldots)$, as discussed in Problem 1 in Chapter 7.

7. Suppose that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is an orthonormal basis in a Hilbert space V over $\mathbb{C}$, and that $(c_n)_{n\in\mathbb{N}}$ is a fixed bounded sequence of complex numbers. Consider the bounded linear operator $T : V \to V$ such that $T(\mathbf{x}_n) = c_n\mathbf{x}_n$ for every $n \in \mathbb{N}$, discussed in Problem 3 in Chapter 7. Find the adjoint operator $T^* : V \to V$.

8. Suppose that V and W are Hilbert spaces over $\mathbb{F}$, and that $T \in B(V,W)$. Suppose also that $R(T)$ and $R(T^*)$ denote respectively the range of the linear transformations $T : V \to W$ and $T^* : W \to V$.
   a) Show that $\langle \mathbf{x}, \mathbf{z}\rangle = 0$ for every $\mathbf{x} \in \ker(T)$ and $\mathbf{z} \in R(T^*)$.
   b) Deduce that $\ker(T) \subseteq (R(T^*))^\perp$.
   c) Show that $\langle T(\mathbf{u}), T(\mathbf{u})\rangle = 0$ for every $\mathbf{u} \in (R(T^*))^\perp$.
   d) Deduce that $(R(T^*))^\perp \subseteq \ker(T)$.
   e) It follows from parts (b) and (d) that $\ker(T) = (R(T^*))^\perp$. Use this and Problem 1 to show that $\ker(T^*) = (R(T))^\perp$.

9. Suppose that V is a Hilbert space over $\mathbb{F}$. Suppose further that $T \in B(V)$ is invertible, so that $TT^{-1} = T^{-1}T = I$, where $I \in B(V)$ is the identity linear operator.
   a) Show that $I^* = I$.
   b) By studying the adjoint of the equation $TT^{-1} = T^{-1}T = I$, show that T^* is invertible, with inverse $(T^*)^{-1} = (T^{-1})^*$.

10. Suppose that V is a Hilbert space over $\mathbb{F}$. Suppose further that $\mathfrak{H}$ is the subset of all Hermitian operators in $B(V)$.
    a) Show that $cT + aS \in \mathfrak{H}$ for every $c, a \in \mathbb{R}$ and $T, S \in \mathfrak{H}$.
    b) Show that $\mathfrak{H}$ is a closed subset of $B(V)$.
    [HINT: Use Problem 4.]

11. Suppose that V is a Hilbert space over $\mathbb{F}$, and that $T \in B(V)$.
    a) Show that T^*T and TT^* are both Hermitian.
    b) Show that there exist Hermitian $R, S \in B(V)$ such that $T = R + \mathrm{i}S$.

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